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# A Super-Resolution Method Based on the Discrete Cosine Transform

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**SUMMARY** In this paper, a super-resolution method based on the Discrete Cosine Transform (DCT) is proposed for a signal with some frequency damage. If the damage process can be modeled as linear convolution with a type 1 linear phase FIR filter, it is shown that some DCT coefficients of the damaged signal are the same as those of the original signal except for the DCT coefficients corresponding to the frequency damage. From this investigation, the proposed method is provided for the DCTs with four types as expanding the super-resolution method based on the Discrete Fourier Transform (DFT). In addition, two magnification approaches based on the proposed method are described to improve the conventional approach.

***key words:*** *super-resolution method, discrete cosine transform, simultaneous linear equations*

## 1. Introduction

Methods for estimating some frequency components damaged due to sampling or other processes are referred to as super-resolution methods in the frequency domain. A principle of the super-resolution methods was shown by Harris [1]. Based on this principle, many super-resolution methods have been explored [2]–[12]. In particular, the Gerchberg-Papoulis (GP) iterative method [2], [3], which is a representative super-resolution method, was proposed as the iterative method in both the time and frequency domains. In the articles [5]–[7], it was shown that the GP-iterative method is equivalent to solving simultaneous linear equations using Jacobi's method. These equations are derived from the relationship between signals in the time domain and their Discrete Fourier Transforms (DFT). Besides, other super-resolution methods are developed as techniques for solving the linear equations formulated from the GP-iterative method. From this point of view, some convergence conditions for the super-resolution methods are also shown [9]–[11]. However, it is necessary to use complex values even when signals are real.

To avoid this problem, the GP-iterative method with the Discrete Cosine Transform (DCT) was applied to the image magnification [12]. Although this method is reduced to real value operations, the error between the original signal and the magnified signal diverges as the number of iteration increases [12]. Therefore, we formulate a new super-resolution method based on the DCT and propose two magnification approaches. The convergence of the error between the original signal and the magnified signal is ensured using the proposed approaches.

In this paper, we first discuss the formulation of a super-resolution method in the DCT domain for a signal with some frequency damage. Firstly, we investigate the DCT of a signal damaged in the high frequency components. We show that, if the damage process can be modeled as linear convolution with a type 1 linear phase FIR filter, some DCT coefficients of a damaged signal are the same as those of an original signal except for the DCT coefficients corresponding to the high frequency components. From this investigation, a new super-resolution method based on the DCT is proposed. The proposed method is suitable for the DCTs with four types. Next, we consider applying the proposed method to the signal magnification. From the results, we show two limitations of the conventional approaches [8], [12], i.e., the one concerns with the applicable type of the DCTs and the other one is caused from the implementation procedure of the conventional approach. These limitations can be removed by two magnification approaches based on the proposed method. In this paper, althogh we consider a super-resolution method for one-dimensional signals, it can be extended for two-dimensional signals such as images.

In Sect. 2, we review the super-resolution method based on the DFT. Next, in Sect. 3, we explain about the damage process for the formulation of a super-resolution method in the DCT domain. Then, in Sect. 4 we propose a super-resolution method based on the DCT. Besides, in Sect. 5, we introduce two approaches based on the proposed method for the signal magnification. Finally, in Sect. 6, we show the result of computer simulations to show the validity of the proposed method.

## 2. The DFT-Based Method [4]–[9]

Before discussing the proposed method, we explain the super-resolution method based on the DFT, called the DFT-based method.

Let us consider reconstructing an original signal $y(n)$ from an observed signal $d(n)$. We assume that the high frequency components of the observed signal $d(n)$ are damaged in the frequency interval

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($\omega_k/2<\omega<2\pi-\omega_k/2$). The DFT-based method is executed under two restricted conditions as follows.

- It is known that the number of the original signal $y(n)$ is $M$.
- Some frequency components of the observed signal $d(n)$ in the interval ($0\leqq\omega\leqq\omega_k/2, 2\pi-\omega_k/2\leqq\omega\leqq 2\pi$) are the same as those of the original signal $y(n)$.

Moreover, this method uses the relationship between the $P$-periodic signals $y_P(n)$ and $d_P(n)$ which are generated from the original signal $y(n)$ and the observed signal $d(n)$, respectively. Then, the length of the period $P$ is chosen as satisfying the inequality

$$P \geqq \left\lfloor \frac{2\pi-\omega_k}{2\pi}P \right\rfloor + M + 1, \qquad (1)$$

where $M$ is the number of the original signal and $2\pi-\omega_k$ is the band width to be reconstructed, and the function $\lfloor c \rfloor$ means the largest integer less than or equal to $c$. Equation (1) is a convergence condition for the super-resolution methods [9]–[11].

The DFT-based method provides the estimated values of the desired DFT coefficients by solving simultaneous linear equations. These equations are formulated from comparing the DFT $Y_P(k)$ of $y_P(n)$ with the DFT $D_P(k)$ of $d_P(n)$, and the coefficient matrix of these equations consists of the sub-matrices of the IDFT matrix. As a result, recovering the damaged frequency components is reduced to solving the simultaneous linear equations, and the original signal can be reconstructed by using these estimated components. Moreover, we can reconstruct not only the high frequency components but also arbitrary frequency components by applying the DFT-based method. However, the DFT-based method has a weak point that the coefficient matrix consists of complex values even when the signal is real.

## 3. DCT of a Signal with Some Damage in the Frequency Domain

In Sect. 2, we summarized the DFT-based method for a signal with some frequency damage. Next, we discuss the DCT of the same signal with some frequency damage. If the damage process can be modeled as linear convolution with a type 1 linear phase FIR filter, it can be shown that some DCT coefficients of the observed signal are the same as those of the original signal as well as the DFT-based method. A linear phase FIR filter as a damage model, which is also used in the conventional methods [2]–[11], means that damaged signals do not include phase distrotion. Moreover, the condition of type 1 allows us to do a super-resolution method based on the relationship between time and DCT domains as shown in this paper.

In the following, we study the relationship between the DCT and the DFT, and explain a constraint for the damage process.

**Table 1** Symmetric-periodic sequence, $N = 2R$.

| | Symmetric-periodic sequences $\widehat{x(n)}$ |
|---|---|
| WSWS | $\widehat{x(n)} = \begin{cases} x(n) & n = 0, \cdots, R \\ x(N-n) & n = R+1, \cdots, N-1 \end{cases}$ |
| HSHS | $\widehat{x(n)} = \begin{cases} x(n) & n = 0, \cdots, R-1 \\ x(N-1-n) & n = R, \cdots, N-1 \end{cases}$ |
| WSWA | $\widehat{x(n)} = \begin{cases} x(n) & n = 0, \cdots, R-1 \\ 0 & n = R \\ -x(N-n) & n = R+1, \cdots, N-1 \end{cases}$ |
| HSHA | $\widehat{x(n)} = \begin{cases} x(n) & n = 0, \cdots, R-1 \\ -x(N-1-n) & n = R, \cdots, N-1 \end{cases}$ |

### 3.1 Relationship Between the DCT and the DFT [14]

It is known that the DCTs with four types can be defined as special cases of the Generalized Discrete Fourier Transform (GDFT) [14]. The entry at row $n$ and column $k$ of the GDFT matrix $\boldsymbol{G}_{a,b}$ of order $N$ is given by

$$[G_{a,b}]_{nk} = \exp\left(-j\frac{2\pi}{N}(n+a)(k+b)\right), \qquad (2)$$
$$0 \leqq n, k \leqq N-1$$

where $a$ and $b$ are real numbers. When $a = b = 0$, $\boldsymbol{G}_{a,b}$ corresponds to the DFT matrix.

From a finite sequence $x(n), n = 0, 1, \cdots, R-1$, we can generate a symmetric-periodic sequence (SPS) $\widehat{x(n)}$ with period $N = 2R$ as shown in Table 1. When computing $N$-point GDFT ($\boldsymbol{G}_{a,b}$) of this sequence $\widehat{x(n)}$, the independent $R$-points coefficients of the GDFT correspond to the DCT coefficients of $x(n)$. Table 2 summarizes the relationship between the GDFT and the DCTs with four types, and Table 1 shows types of the symmetric-periodic sequences $\widehat{x(n)}$.

### 3.2 DCT of a Signal with Some Frequency Damage

Now we consider an original signal $y_P(n)$ and an observed signal $d_P(n)$ as shown in Fig. 1, where the representation of 'high' with '×' means that the high frequency components of the observed signal $d_P(n)$ are damaged. The observed signal $d_P(n)$ is damaged in the frequency interval ($\omega_k/2<\omega<2\pi-\omega_k/2$) as well as that of the DFT-based method. The number $P$ of these signals is chosen as satisfying the condition in Eq. (1). As shown in Eq. (3), we assume that the observed signal $d_P(n)$ is represented as linear convolution of the original signal $y_P(n)$ with a type 1 linear phase FIR lowpass filter $h(n)$ of order $\mathcal{N}_h$.

$$d_P(n) = y_P(n) * h(n), \qquad (3)$$

where '$*$' denotes linear convolution, $h(n)$ has symmetric impulse response and $\mathcal{N}_h$ is even. Moreover, we assume that the relationship between order $\mathcal{N}_h$ of the filter $h(n)$ and the number $P$ is organized as follows:

**Table 2** The relationship the DCT and the GDFT.

| DCT | symmetric extentions | $G_{a,b}$ |
|---|---|---|
| DCT-I | WSWS | $G_{0,0}$ |
| DCT-II | HSHS | $G_{0,\frac{1}{2}}$ |
| DCT-III | WSWA | $G_{\frac{1}{2},0}$ |
| DCT-IV | HSHA | $G_{\frac{1}{2},\frac{1}{2}}$ |

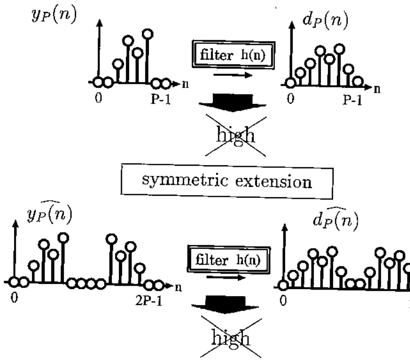


**Fig. 1** The relationship between the DCT and the DFT.

$$P = z_\ell + z_r + M, \quad z_\ell, z_r \geqq N_h/2, \tag{4}$$

where $z_\ell$ and $z_r$ are the number of zero values of $y_P(n)$ in the left and the right side. Under the condition in Eq. (4), the linear convolution gives the same sequence as the circular convolution with the period $P$.

Next, from the original signal $y_P(n)$, for example, we generate a symmetric-periodic sequence $\widetilde{y_{2P}(n)}$ that is called HSHS (*half-sample symmetry half-sample symmetry*) [14] sequence with period $2P$ in Table 1. Well, let us consider the following equation,

$$\widetilde{d_{2P}(n)} = \widetilde{y_{2P}(n)} * h(n). \tag{5}$$

If $h(n)$ is a type 1 linear phase FIR filter, it is known that $\widetilde{d_{2P}(n)}$ is also an HSHS sequence as well as $\widetilde{y_{2P}(n)}$ [14]. Then, $\widetilde{d_{2P}(n)}$ is equivalent to the HSHS sequence generated from $d_P(n)$ (see Fig. 1). While, if $h(n)$ is a linear phase FIR filter of other types, it is impossible for $\widetilde{d_{2P}(n)}$ to keep the HSHS property of the SPS.

From the relationship between Eq. (3) and Eq. (5), we can see that the GDFT of the HSHS sequence $\widetilde{d_{2P}(n)}$ is damaged in the interval $(\omega_k/2<\omega<2\pi-\omega_k/2)$ as well as the DFT of $d_P(n)$. As shown in Table 2, the type-II DCT coefficients are obtained by computing the GDFT of the HSHS sequence. Therefore, the DCT coefficients $D_P(k)$ of $d_P(n)$ are also damaged in the interval $(\lfloor \frac{\omega_k}{2\pi} P \rfloor \leqq k \leqq P-1)$. Moreover, we should note that both the DCT coefficients of $y_P(n)$ and $d_P(n)$ can be represented as the same type-II DCT coefficients under a type 1 linear phase FIR filter, because both $\widetilde{y_{2P}(n)}$ and $\widetilde{d_{2P}(n)}$ in Eq. (5) are the same HSHS sequences. As a result, some DCT coefficients of $d_P(n)$ in the interval $(0 \leqq k < \lfloor \frac{\omega_k}{2\pi} P \rfloor)$ are the same as those of $y_P(n)$.

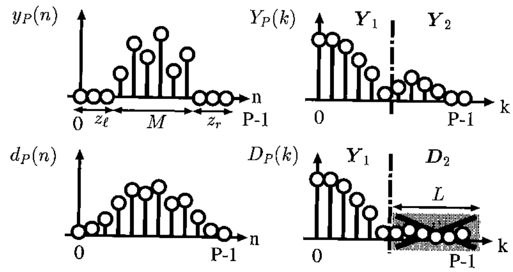


**Fig. 2** The original signal $y_P(n)$ and the observed signal $d_P(n)$.

For the DCT of other types, we can derive the similar results as well as the result for the DCT-II. That is, some DCT coefficients of the observed signal are the same as those of the original signal under a type 1 linear phase FIR filter. The difference is to discuss other types of the SPS, such as WSWS (*whole-sample symmetry whole-sample symmetry*), WSWA (*whole-sample symmetry whole-sample antisymmetry*) and HSHA (*half-sample symmetry half-sample antisymmetry*), instead of HSHS.

## 4. The Proposed Method

Based on the above discussion, we formulate a super-resolution method in the DCT domain for a signal which is damaged in the high frequency components. In this section, we propose a super-resolution method based on the DCT, referred to as the DCT-based method. For convenience of the discussion, we consider only type-II DCT.

### 4.1 Restricted Conditions

As shown in Fig. 2, $M$ points on $P$ points of the original signal $y_P(n)$ are not guaranteed zero values. The observed signal $d_P(n)$ is damaged in the DCT coefficients corresponding to the high frequency components. And $Y_P(k)$ and $D_P(k)$ are the DCT of $y_P(n)$ and $d_P(n)$, respectively. Then we assume that the number of the damaged DCT coefficients, $Y_P(k) \neq D_P(k)$, is $L$ points.

The DCT-based method is executed under two restricted conditions as follows.

- We know the regions not guaranteed $y_P(n)=0$ and its number $M$.
- The DCT coefficients corresponding to the observed signal $d_P(n)$ in the interval $(0 \leqq k \leqq P-L-1)$ are the same as those of the original signal $y_P(n)$.

Also, the number $P$ is chosen as satisfying the inequality

$$P - M = z_\ell + z_r \geqq L. \tag{6}$$

Equation (6) corresponds to Eq. (1) and can be regarded as a convergence condition for the super-resolution method in the DCT domain.

### 4.2 DCT Expression of Signals

The relationship between the original signal $y_P(n)$ and its DCT $Y_P(k)$ is shown in matrix form as follows

$$\boldsymbol{y}_P = \boldsymbol{Ci}_P^{\mathrm{II}} \boldsymbol{Y}_P, \tag{7}$$

where $\boldsymbol{y}_P$, $\boldsymbol{Y}_P$ are the vectors of $y_P(n)$ and $Y_P(k)$, respectively, and the entry at row $n$ and column $k$ of the IDCT matrix $\boldsymbol{Ci}_P^{\mathrm{II}}$ of order $P$ is given by [13]

$$\left[\boldsymbol{Ci}_P^{\mathrm{II}}\right]_{nk} = \sqrt{\frac{2}{P}} O_k \cos\left(\frac{(n+\frac{1}{2})k\pi}{P}\right), \quad 0 \leqq n, k \leqq P-1 \tag{8}$$

$$O_k = \begin{cases} 1/\sqrt{2} & k = 0 \\ 1 & k \neq 0 \end{cases}. \tag{9}$$

We also obtain the relation between the observed signal $d_P(n)$ and its DCT $D_P(k)$ in a manner,

$$\boldsymbol{d}_P = \boldsymbol{Ci}_P^{\mathrm{II}} \boldsymbol{D}_P, \tag{10}$$

where $\boldsymbol{d}_P$, $\boldsymbol{D}_P$ and $\boldsymbol{Ci}_P^{\mathrm{II}}$ are the vectors of $d_P(n)$, $D_P(k)$ and the IDCT matrix in Eq. (8), respectively.

From the second of the restricted conditions, some DCT coefficients corresponding to low frequency components of $D_P(k)$ are the same as those of $Y_P(k)$. As a result, the vectors of the DCT coefficients $\boldsymbol{Y}_P$ and $\boldsymbol{D}_P$ are represented as

$$\boldsymbol{Y}_P = \left[\frac{\boldsymbol{Y}_1}{\boldsymbol{Y}_2}\right], \quad \boldsymbol{D}_P = \left[\frac{\boldsymbol{Y}_1}{\boldsymbol{D}_2}\right]. \tag{11}$$

Then, as shown in Fig. 2, $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$ and $\boldsymbol{D}_2$ are defined as follows

$$\begin{aligned} \boldsymbol{Y}_1 &= [Y_P(0), \cdots, Y_P(P-L-1)]^T \\ \boldsymbol{Y}_2 &= [Y_P(P-L), \cdots, Y_P(P-1)]^T \\ \boldsymbol{D}_2 &= [D_P(P-L), \cdots, D_P(P-1)]^T \end{aligned}$$

where subscript $T$ denotes vector transpose. As shown in Eq. (11), the vectors of the DCT coefficients $\boldsymbol{Y}_P$ and $\boldsymbol{D}_P$ are the same except for $\boldsymbol{Y}_2$ and $\boldsymbol{D}_2$ corresponding to the high frequency components. In the DCTs with other types, we only replace $\boldsymbol{Ci}_P^{\mathrm{II}}$ in Eqs. (7), (10) with their corresponding the IDCT matrix.

### 4.3 Simultaneous Linear Equations

We derive that recovering the damaged DCT coefficients is reduced to solving simultaneous linear equations. As shown in Fig. 2, owing to the damaged DCT vector $\boldsymbol{D}_2$,

$$y_P(n) = 0, \quad \begin{cases} 0 \leqq n \leqq z_\ell - 1 \\ z_\ell + M - 1 \leqq n \leqq P - 1 \end{cases} \tag{12}$$

cannot be guaranteed. To make the damaged vector $\boldsymbol{D}_2$ equal to $\boldsymbol{Y}_2$, the following equation has to be satisfied:

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}_1^{\mathrm{II}} & \boldsymbol{C}_2^{\mathrm{II}} \end{bmatrix} \left[\frac{\boldsymbol{Y}_1}{\boldsymbol{Y}_2}\right], \tag{13}$$

where $\boldsymbol{C}_i^{\mathrm{II}} (i = 1, 2)$ are sub-matrices of $\boldsymbol{Ci}_P^{\mathrm{II}}$ in Eq. (8), and the entry at row $n$ and column $k$ of these matrices is given by, respectively,

$$\left[\boldsymbol{C}_i^{\mathrm{II}}\right]_{nk} = \begin{cases} \left[\boldsymbol{Ci}_P^{\mathrm{II}}\right]_{nm_i} & 0 \leqq n \leqq z_\ell - 1 \\ \left[\boldsymbol{Ci}_P^{\mathrm{II}}\right]_{(n+M)m_i} & z_\ell \leqq n \leqq P - M - 1 \end{cases} \tag{14}$$

$$m_i = \begin{cases} k, & 0 \leqq k \leqq P-L-1 \quad (i = 1) \\ P - L + k, & 0 \leqq k \leqq L - 1 \quad (i = 2) \end{cases}$$

Now, we note that $\boldsymbol{C}_1^{\mathrm{II}}$ and $\boldsymbol{C}_2^{\mathrm{II}}$ are known matrices and $\boldsymbol{Y}_1$ is a known vector of the DCT coefficients. Thus, we make simultaneous linear equations from Eq. (13) as

$$\boldsymbol{C}_2^{\mathrm{II}} \boldsymbol{Y}_2 = -(\boldsymbol{C}_1^{\mathrm{II}} \boldsymbol{Y}_1). \tag{15}$$

Therefore, the original signal can be reconstructed by solving Eq. (15). However, in general, the system of these linear equations is rank-deficient. That is, the number of linearly independent equations is less than the number of unknowns or the number of equations. Thus, the inverse matrix of $\boldsymbol{C}_2^{\mathrm{II}}$ does not exist. Equation (15) is solved under the least-squares criteria as well as that of the DFT-based method [7]. As a result, we can get the optimal solution $\widetilde{\boldsymbol{Y}}_2$. The original signal $y_P(n)$ can be reconstructed by using $\widetilde{\boldsymbol{Y}}_2$.

## 5. Magnification with the DCT-Based Method

In this section, we consider applying the super-resolution method to the signal magnification. The image magnification based on the super-resolution method has been reported so far [8], [12]. In the article [12], the GP-iterative method with the DCT has been applied to the image magnification. We first show the conventional procedure, and investigate whether the magnified signal can be regarded as the reconstructed signal. Next, we propose two procedures to improve the conventional magnification procedure. In the following, from the discussion in Sect. 3, we discuss the signal magnification under a type 1 linear phase FIR filter as a damage model.

### 5.1 The Conventional Procedure [12]

First, we show the conventional procedure. We assume that the number of an original signal $y(n)$ is $M$ and an observed signal $d(n)$ is the decimated signal of the original signal, that is, the original signal $y(n)$ is band-limited to the region $|\omega| < \pi/\mathcal{D}$ and down-sampled by the down-sampled factor $\mathcal{D}$. Then, the number of $d(n)$

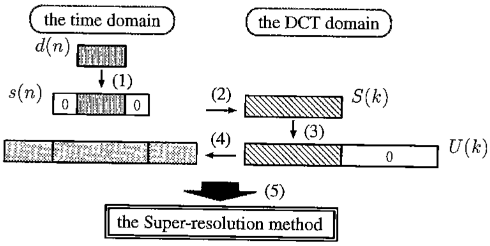


**Fig. 3** The conventional procedure.

is $K(=M/\mathcal{D})$. The conventional procedure is executed as follows (see Fig. 3).

**Step 1:** Obtain the $P_K$ points signal $s(n)$ from the $K$ points observed signal $d(n)$ as

$$s(n)=\begin{cases}0 & 0\leqq n\leqq z_\ell-1\\ d(n-z_\ell) & z_\ell\leqq n\leqq z_\ell+K-1\\ 0 & z_\ell+K\leqq n\leqq P_K-1\end{cases}. \quad (16)$$

**Step 2:** Compute the $P_K$ points DCT-II coefficients $S(k)$ of $s(n)$.

**Step 3:** Extend the DCT coefficients $S(k)$ to the $\mathcal{U}P_K$ points DCT coefficients $U(k)$ as

$$U(k)=\begin{cases}S(k) & 0\leqq k\leqq P_K-1\\ 0 & P_K\leqq k\leqq \mathcal{U}P_K-1\end{cases}, \quad (17)$$

where $\mathcal{U}$ means the extension ratio.

**Step 4:** Compute the $\mathcal{U}P_K$ points IDCT-II $u(n)$ of $U(k)$.

**Step 5:** Obtain $\widetilde{u(n)}$ by applying the super-resolution method for $u(n)$ under two restricted conditions provided in Sect. 4.1. Take the non-zero $M$ points magnified signal $\widetilde{y(n)}$ from the $\mathcal{U}P_K$ points signal $\widetilde{u(n)}$ as

$$\widetilde{y(n)}=\tilde{u}(n+\mathcal{U}z_\ell) \quad 0\leqq n\leqq M-1. \quad (18)$$

We apply this conventional procedure to the signal magnification. We show the result of computer simulation in Fig. 4 under the conditions:

- The original signal $y(n)$ is taken $M$=16 points from the 110th row of the "Barbara" image.
- The observed signal $d(n)$ is the decimated version of the original signal. That is, the original signal is band-limited to the region $|\omega|<\pi/2$ and down-sampled by the down-sample factor $\mathcal{D}=2$.
- The number of the observed signal $d(n)$, zero values in Eq. (16) and the signal $s(n)$ provided in **Step 1** are $K$=8, $z_\ell+z_r$=8 and $P_K=16$, respectively.
- The extension ratio is $\mathcal{U}=2$. Thus, the number of the signal $u(n)$ is $\mathcal{U}P_K$=32.
- We apply the DCT-based method, which solves Eq. (15) by the conjugate gradient method, for $u(n)$.

As the reference of comparison we use the equation defined as follows,

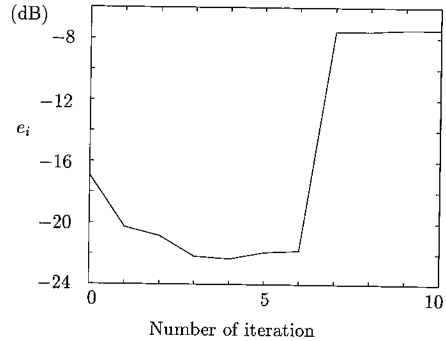


**Fig. 4** Comparison of the error between the original signal and the reconstructed signal obtained by the conventional procedure.

$$e_i=10\,log_{10}\frac{\|Y(k)-\widetilde{Y_i(k)}\|}{\|Y(k)\|}, \quad (19)$$

where $Y(k)$ and $\widetilde{Y_i(k)}$ are the DCT coefficients of the original signal $y(n)$ and its estimated signal after $i$th iteration, respectively, and $\|\cdot\|$ denotes a square norm.

From the convergence characteristics in Fig. 4, we see that the error between the original signal $y(n)$ and the reconstructed signal $\widetilde{y(n)}$ decreases during the first iterations and tends to diverge as the number of iteration increases. This tendency has been also described in the article [12]. Next, we show two reasons for causing this divergence.

## 5.2 Consideration on the Conventional Procedure

### *A. the reason-1*

For the signal magnification, an observed signal $d(n)$ can be regarded as the decimated version of an original signal $y(n)$. That is, an original signal $y(n)$ is band-limited by a lowpass filter, which is a type 1 linear phase FIR filter, with cutoff frequency $\pi/\mathcal{D}$ and down-sampled by the down-sample factor $\mathcal{D}$. Thus, when applying the DCT-based method to the signal magnification, it is necessary that the DCT coefficients corresponding to low frequency components of $d(n)$ are the same as those of $y(n)$. For the DCTs with four types, we have studied the relation between the DCT coefficients of $y(n)$ and those of $d(n)$ [15] (see Appendix A). As a result, in the type-II and -IV cases, the DCT coefficients corresponding to low frequency components of $y(n)$ and $d(n)$ are not the same even if we employ an ideal lowpass filter, on the other hand those are the same in the type-I and -III cases. Because computing the DCT of type-II or - IV needs a half time-shift as shown in Eq. (8).

### *B. the reason-2*

Moreover, even if we execute this conventional procedure with the DCT of type-I or -III, we can not regard the magnified signal as the reconstructed signal. In the

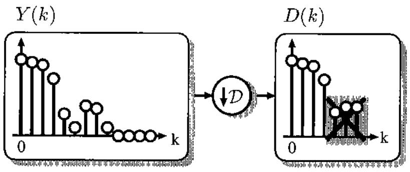


**Fig. 5** The DCT coefficients $Y(k)$ of the original signal and the DCT coefficients $D(k)$ of the observed signal ($\mathcal{D}=2$).

conventional procedure, we can obtain the up-sampled version of $s(n)$ provided in **Step 1** as a reconstructed signal. Because the up-sampled signal of $s(n)$ is also satisfied two restricted conditions in Sect. 4.1. Thus, it is impossible to magnify a signal by using this procedure. We have verified this result by doing computer simulations [15].

Therefore, we think that the above reasons cause the divergence of the convergence characteristics in Fig. 4. Next, we propose two magnification procedures to improve the conventional procedure.

### 5.3 The Proposed Procedure

We propose two procedures with the DCT of type-I or -III for the signal magnification. For convenience, we discuss the signal magnification for the up-sample factor $\mathcal{U}=2$. We assume that an original signal $y(n)$ and an observed signal $d(n)$ are signals as shown in Fig. 5. The $M$ points original signal $y(n)$ is band-limited, but is not necessarily band-limited to the region $|\omega|<\pi/2$. The observed signal $d(n)$ is the down-sampled signal of $y(n)$ for the down-sample factor $\mathcal{D}=2$. In the following, we show two magnification procedures for $d(n)$.

*A. The Proposed Procedure-1*

**Step 1:** Up-sample the observed signal $d(n)$ by the up-sample factor $\mathcal{U}$ ($\mathcal{U}$=2) and obtain the $M$ points signal $v(n)$ as

$$v(n)=\begin{cases} d(n/\mathcal{U}) & n=0,\pm\mathcal{U},\pm 2\mathcal{U},\cdots \\ 0 & \text{otherwise.}\end{cases} \tag{20}$$

**Step 2:** Do linear convolution of $v(n)$ with a filter $h_u(n)$ which is a lowpass filter band-limited to the region $|\omega|<\pi/\mathcal{U}$, and obtain the signal $w(n)$ as

$$w(n)=v(n)*h_u(n). \tag{21}$$

**Step 3:** Select the number $P$ as satisfied the condition in Eq. (6) and obtain the $P$ points signal $w_P(n)$ from $w(n)$.
**Step 4:** Compute the $P$ points DCT coefficients $W_P(k)$ of $w_P(n)$.
**Step 5:** Apply the DCT-based method for the signal $w_P(n)$ under the following two conditions.

1. It is known that the number of the original signal $y(n)$ is $M$.
2. The DCT coefficients corresponding to low frequency components of $w_P(n)$ are the same as those of the signal $y_P(n)$. Then, the signal $y_P(n)$ is $P$ points signal extended from the original signal $y(n)$ as

$$y_P(n)=\begin{cases} 0 & 0\leqq n\leqq z_\ell-1 \\ y(n-z_\ell) & z_\ell\leqq n\leqq z_\ell+M-1 \\ 0 & z_\ell+M\leqq n\leqq P-1\end{cases}. \tag{22}$$

The signal magnification is completed by executing the above procedure. However, the accuracy of the obtained signal depends on the property of the filter $h_u(n)$ in Eq. (21). Thus, it is difficult to select the filter $h_u(n)$ employed in **Step 2** in order to guarantee the restricted condition **2.**. Next, we propose another magnification procedure without any filters.

*B. The Proposed Procedure-2*

**Step 1:** Execute the same as the procedure-1.
**Step 2:** Compute the $M$ points DCT coefficients $V(k)$ of $v(n)$.
**Step 3:** Apply the DCT-based method for the signal $v(n)$ under the following three conditions.

1. The samples of $v(n)$ except for zero values, $d(n/\mathcal{U})$, are the same as those of the original signal $y(n)$.
2. The DCT coefficients corresponding to low frequency components of $v(n)$ are the same as those of the original signal $y(n)$.
3. We reduce the DCT coefficients corresponding to higher frequency components than the damaged frequency components to zero values.

## 6. Simulation Results

In order to verify the validity of the proposed method, we did computer simulations for two cases.

First, we apply the DCT-based method provided in Sect. 4 to reconstructing an original signal from an observed signal. The conditions of the simulations are as follows:

- The original signal $y_P(n)$ is a signal added zero values in the left and the right sides to an $M$=16 points signal taken from the 110th row of the "Barbara' image. $z_\ell$ and $z_r$, the number of zero values of $y_P(n)$ in the left and the right sides, are $z_\ell$=$z_r$=8. The number of $y_P(n)$ is $P$=32.
- The observed signal is damaged in the interval ($\pi/2<\omega<3\pi/2$). Thus, the number of the damaged DCT coefficients is $L$=16.
- The DCT-based method is executed with the DCT of type-II under two restricted conditions provided in Sect. 4.1, and the simultaneous linear equations are solved by the conjugate gradient method.

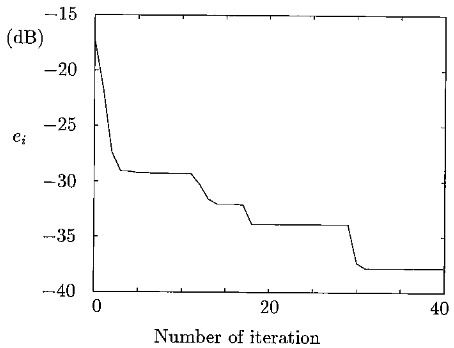


**Fig. 6** Comparison of the error between the original signal and the reconstructed signal obtained by applying the proposed method for the observed signal. The observed signal is damaged in the DCT coefficients corresponding to high frequency components.

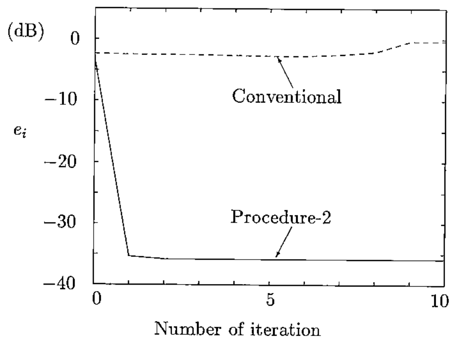


**Fig. 7** Comparison of the result of applying the procedure-2 and that of applying the conventional procedure.

We use Eq. (19) as the reference of comparison. We result in the convergence characteristics as shown in Fig. 6. From this result, we can see that the original signal $y(n)$ is reconstructed.

Next, we verify the validity of applying the proposed procedure-2 provided in Sect. 5 to the signal magnification. Figure 7 shows the result of applying the procedure-2 and that of applying the conventional procedure provided in Sect. 5.1 under the conditions:

- The original signal $y(n)$ is taken from the 110th row of the "Barbara" image and band-limited by a lowpass filter with cutoff frequency $2\pi/3$. The number of $y(n)$ is 17.
- The observed signal $d(n)$ is the down-sampled signal of $y(n)$ for the down-sample factor $\mathcal{D}$=2. Thus, the number of $d(n)$ is 9.
- The proposed procedure-2 with the 17-point DCT-I and the conventional procedure are executed for the up-sample factor $\mathcal{U}$=2, respectively, and the simultaneous linear equations are solved by the conjugate gradient method.
- We use Eq. (19) as the reference of comparison.

From Fig. 7, we can verify that the proposed procedure-2 provides the convergence of the error between the original signal and the magnified signal, while the conventional procedure not. As well as the DCT of type-I, we can derive the similar result for the proposed procedure-2 with the DCT of type-III.

## 7. Conclusion

In this paper, we proposed the super-resolution method based on the DCT for a signal with some frequency damage. For that, we discussed the DCT of the signal with some frequency components. When the damage process can be modeled as linear convolution with a type 1 linear phase FIR filter, we showed that some DCT coefficients of an observed signal are the same as those of an original signal except for the DCT coefficients corresponding to the frequency damage. We proposed the DCT-based method for the DCTs with four types. Besides, we considered applying the proposed method to the signal magnification. We showed two limitations for the conventional approach. Then we proposed two magnification approaches to improve the conventional approach. Finally, we showed the result of computer simulations. From the results, we verified that the proposed method achieves to reconstruct an original signal.

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## Appendix : The Relationship Between the DCT Coefficients of $y(n)$ and Those of $d(n)$

The relationship between the $M+1$ points original signal $y(n)$ and its DCT-I $Y(k)$ is shown as follows

$$y(n) = \sqrt{\frac{2}{M}} O(n) \sum_{k=0}^{M} O(k) Y(k) \cos\left(\frac{nk\pi}{M}\right), \quad n,k = 0,1,\cdots,M \qquad (A\cdot 1)$$

$$O(k) = \begin{cases} 1/\sqrt{2} & k = 0, M \\ 1 & \text{otherwise} \end{cases}. \qquad (A\cdot 2)$$

The $K(=M/\mathcal{D})+1$ points observed signal $d(n)$, which is the down-sampled signal of $y(n)$ for the down-sample factor $\mathcal{D}$, can be represented as

$$d(m) = y(\mathcal{D}m). \qquad m = 0,1,\cdots,K \qquad (A\cdot 3)$$

By using Eq. (A· 1) in Eq. (A· 3), we obtain

$$d(m) = \sqrt{\frac{2}{M}} O(\mathcal{D}m) \sum_{k=0}^{M} O(k) Y(k) \cos\left(\frac{\mathcal{D}mk\pi}{\mathcal{D}K}\right). \quad m = 0,1,\cdots,K \quad k = 0,1,\cdots,M \qquad (A\cdot 4)$$

Equation (A· 4) can be shown in matrix form as follows

$$\boldsymbol{d}_{K+1} = \sqrt{\frac{2}{M}} \boldsymbol{O}_{K+1} \boldsymbol{C}_{K+1,M+1} \boldsymbol{O}_{M+1} \boldsymbol{Y}_{M+1}, \qquad (A\cdot 5)$$

where $\boldsymbol{d}_{K+1}$ and $\boldsymbol{Y}_{M+1}$ are the vectors of $d(n)$ and $Y(k)$, respectively. $\boldsymbol{O}_{K+1}$ and $\boldsymbol{O}_{M+1}$ are the $K+1 \times K+1$ diagonal matrix of Eq. (A· 2) and the $M+1 \times M+1$ one. $\boldsymbol{C}_{K+1,M+1}$ is a matrix of order $K+1 \times M+1$.

Now we consider $\boldsymbol{C}_{K+1,M+1}$ in Eq. (A· 5). We note that index $k$ can be rewritten as

$$k = \left\lfloor \frac{k}{K} \right\rfloor K + ((k))_K, \qquad (A\cdot 6)$$

where the function $\lfloor c \rfloor$ means the largest integer less than or equal to $c$ and $((k))_K$ is the remainder of the division of $k$ by $K$. Substituting Eq. (A· 6) into the basis function in Eq. (A· 4), we obtain

$$\cos\left(\frac{mk\pi}{K}\right) = \cos\left(\left\lfloor \frac{k}{K} \right\rfloor m\pi + \frac{m((k))_K \pi}{K}\right). \qquad (A\cdot 7)$$

If $\lfloor k/K \rfloor$ is even, Eq. (A· 7) can be shown as

$$\cos\left(\left\lfloor \frac{k}{K} \right\rfloor m\pi + \frac{m((k))_K \pi}{K}\right) = \cos\left(\frac{m((k))_K \pi}{K}\right). \qquad (A\cdot 8)$$

We can see that Eq. (A· 8) is equivalent to the $K+1$-point DCT matrix. If $\lfloor k/K \rfloor$ is odd, then

$$\cos\left(\left\lfloor \frac{k}{K} \right\rfloor m\pi + \frac{m((k))_K \pi}{K}\right) = \cos\left(\frac{m(K - ((k))_K)\pi}{K}\right). \qquad (A\cdot 9)$$

We can see that Eq. (A· 9) is the reversal of the $K+1$-point DCT matrix. In words, Eq. (A· 9) is obtained from the $K+1$-point DCT matrix by renumbering the columns in reverse order. As a result, for $\mathcal{D}$=2, Eq. (A· 5) can be represented with the $K+1$-point DCT matrix as

$$\boldsymbol{d}_{K+1} = \sqrt{\frac{2}{M}} \boldsymbol{O}_{K+1} \boldsymbol{C}^{\mathrm{I}}_{K+1} \boldsymbol{T}_{K+1,M+1} \boldsymbol{O}_{M+1} \boldsymbol{Y}_{M+1}, \qquad (A\cdot 10)$$

where $\boldsymbol{C}^{\mathrm{I}}_{K+1}$ is the $K+1$-point DCT matrix and $\boldsymbol{T}_{K+1,M+1}$ is the $K+1 \times M+1$ matrix as

$$\boldsymbol{T}_{K+1,M+1} = \begin{bmatrix} 1 & 0 & \cdots & & \cdots & 0 & 1 \\ 0 & 1 & & & & 1 & 0 \\ \vdots & & \ddots & & \cdot^{\cdot^{\cdot}} & & \vdots \\ 0 & & & 1 & & & 0 \end{bmatrix}. \qquad (A\cdot 11)$$

From Eq. (A· 10), the DCT coefficients $D(k)$ of $d(n)$ can be shown in matrix form as follows

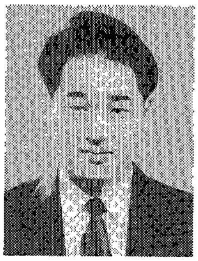
$$\boldsymbol{D}_{K+1} = \frac{1}{\sqrt{\mathcal{D}}} \boldsymbol{O}_{K+1} \boldsymbol{T}_{K+1,M+1} \boldsymbol{O}^{-1}_{M+1} \boldsymbol{Y}_{M+1}. \qquad (A\cdot 12)$$

Thus, we can see that the relationship between $Y(k)$ and $D(k)$ is represented by $\boldsymbol{T}_{K+1,M+1}$. In Eq. (A· 12), we note that, if $\boldsymbol{O}_{K+1}$ and $\boldsymbol{O}_{M+1}$ are the identity matrices, the result is similar to that for the relationship between the DFT coefficients of $y(n)$ and those of $d(n)$.

For the DCT of other types, we can similarly derive the relationship between the DCT coefficients of the original signal and those of the observed signal. For the DCT-III, we can derive a similar result as well as that for the DCT-I. In this case, the relationship between $Y(k)$ and $D(k)$ for the DCT-III is represented by $\boldsymbol{S}$ instead of $\boldsymbol{T}_{K+1,M+1}$ in Eq. (A· 12), where $\boldsymbol{S}$ is a matrix given by

$$\boldsymbol{S} = \begin{bmatrix} 1 & 0 & \cdots & & & \cdots & 0 & 1 \\ 0 & 1 & & & & & 1 & 0 \\ \vdots & & \ddots & & & \cdot^{\cdot^{\cdot}} & & \vdots \\ 0 & & & 1 & 1 & & & 0 \end{bmatrix}. \qquad (A\cdot 13)$$

Both of $\boldsymbol{O}_{K+1}$ and $\boldsymbol{O}_{M+1}$ in Eq. (A· 12) become the identity matrices. Thus, we can see that the result for the DCT-III is equivalent to that for the DFT. On the other hand, for the DCT-II or - IV, the relationship between the DCT coefficients of $y(n)$ and those of $d(n)$ is different from the result for the DFT.



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