A Design Method of Multidimensional Linear-Phase Paraunitary Filter Banks with a Lattice Structure

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***Abstract*—A lattice structure of multidimensional (MD) linear-phase paraunitary filter banks (LPPUFB's) is proposed, which makes it possible to design such systems in a systematic manner. Our proposed structure can produce MD-LPPUFB's whose filters all have the region of support $\mathcal{N}(\boldsymbol{M}\Xi)$, where $\boldsymbol{M}$ and Ξ are the decimation and positive integer diagonal matrices, respectively, and $\mathcal{N}(\boldsymbol{N})$ denotes the set of integer vectors in the fundamental parallelepiped of a matrix $\boldsymbol{N}$. It is shown that if $\mathcal{N}(\boldsymbol{M})$ is reflection invariant with respect to some center, then the reflection invariance of $\mathcal{N}(\boldsymbol{M}\Xi)$ is guaranteed. This fact is important in constructing MD linear-phase filter banks because the reflection invariance is necessary for any linear-phase filter. Since our proposed system structurally restricts both the paraunitary and linear-phase properties, an unconstrained optimization process can be used to design MD-LPPUFB's. Our proposed structure is developed for both an even and an odd number of channels and includes the conventional 1-D system as a special case. It is also shown to be minimal, and the no-DC-leakage condition is presented. Some design examples will show the significance of our proposed structure for both the rectangular and nonrectangular decimation cases.**

***Index Terms*— Lattice structure, linear-phase, multidimensional filter banks, paraunitary.**

I. Introduction

The application of filter banks to data compression, which is known as subband coding (SBC), has been studied as an effective coding scheme in audio and visual communications [1]–[3]. Recently, multidimensional (MD) multirate signal processing has increasingly been used in video processing [4], and interest in MD filter banks has risen.

Multirate filter banks are usually composed of both analysis and synthesis banks. The analysis bank decomposes a signal into different frequency subbands, and the synthesis bank reconstructs the original signal from the subband signals. If the reconstructed signal is identical to the original one, except for delay and scaling, then the analysis–synthesis system is said to be a perfect reconstruction (PR) filter bank. PR filter banks, where filters in the synthesis bank are complex-conjugated spatiotemporal reversals of ones in the analysis bank, are referred to as paraunitary (PU) filter banks [1]–[3]. In most applications, subband signals are processed. For example, they are quantized in the SBC application. Any process for subband signals affects the reconstructed signals, even though the PR system is used. In such a situation, the PU system has the advantage that it guarantees the error energy in the reconstructed signal to be the average of the error energy in the subband signals. This property also allows us to use optimal bit-allocation algorithms [5].

The human visual system is known to be sensitive to phase distortion. Since phase distortion can be avoided by applying filters with a linear-phase (LP) property, it is desirable that all filters composing filter banks have the LP property when the system is applied to image processing. Hence, the LP and PU properties of filter banks are particularly significant for the SBC of images. Several one-dimensional (1-D) linear-phase paraunitary filter banks (LPPUFB's) have been developed so far [3], [6]–[11]. The lattice and the modulation-based structures in particular have received a lot of attention because they enable us to design LPPUFB's in a systematic way, and some of them enable fast implementation.

One-dimensional LPPUFB's can be applied to the construction of MD separable systems. However, MD signals are generally nonseparable, and this approach does not exploit their characteristics effectively. Furthermore, systems that consist of 1-D two-channel filter banks do not have an overlapping solution with LP and PU properties. To overcome these disadvantages, nonseparable MD-LPPUFB's are required. An extension to two dimensions has been presented for modulation-based systems [12], [13], where the design process is very cost efficient because just one prototype filter needs to be designed. A drawback to this, however, is that the classes that are covered are limited.

At the same time, the 1-D lattice structure is known to cover a large class of LPPUFB's [6]–[8], [10], [11]. The structure makes it possible to design 1-D LPPUFB's by characterizing some orthonormal matrices. Early in the 1990's, the lattice structure for two-dimensional (2-D) LPPUFB's had already been discussed by Karlsson and Vetterli [14]. Then, the structures were based on the sophisticated 1-D works so that 2-D LPPUFB's could be designed in a similar way to 1-D ones [15]. The structure discussed by Kovačević and Vetterli [15], however, is for separable decimation with an even number of channels. So far, the extension to multiple dimensions for nonseparable decimation has never been discussed, nor has that for an odd number of channels. In light of these facts, we propose construction of a lattice structure of MD-LPPUFB's so that a large class of MD ones can be designed in a systematic way. Note that the discussion here is not restricted to two dimensions. The results can be applied to any number of dimensions.

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Section II of this paper discusses some properties of MD LP filters, especially with the region of support, and provides the LP condition in polyphase representation. Section III reviews MD filter banks and clarifies the class dealt with in this paper. Section IV details the main topic of this paper, where a lattice structure of MD-LPPUFB's is proposed. Finally, Section V shows some design examples, and conclusions are presented in Section VI.

The following notation is used throughout this paper.

D — Number of dimensions.

$\boldsymbol{z}$ — $D\times 1$ vector that consists of variables in a D-dimensional Z domain, that is, $\boldsymbol{z}=(z_0, z_1, \cdots, z_{D-1})^T$.

$\boldsymbol{z}^{\boldsymbol{m}}$ — product defined by

$$\boldsymbol{z}^{\boldsymbol{m}} = z_0^{m_0} z_1^{m_1} \cdots z_{D-1}^{m_{D-1}} \tag{1}$$

where $\boldsymbol{m}$ is a $D\times$ 1 integer vector, and m_k denotes the kth element of $\boldsymbol{m}$.

$\boldsymbol{z}^{\boldsymbol{M}}$ — $D \times 1$ vector whose dth element is defined by

$$[\boldsymbol{z}^{\boldsymbol{M}}]_d = z_0^{M_{0,d}} z_1^{M_{1,d}} \cdots z_{D-1}^{M_{D-1,d}} \tag{2}$$

where $\boldsymbol{M}$ is a $D \times D$ nonsingular integer matrix, and $M_{k,\ell}$ denotes the kth row and ℓth column element of $\boldsymbol{M}$.

$\boldsymbol{O}, \boldsymbol{o}$ — null matrix and the null column vector, respectively.

$\boldsymbol{I}_M, \boldsymbol{J}_M$ — $M \times M$ identity and counter-identity matrices, respectively. When the size is obvious or not of interest, the subscript M is omitted.

$\boldsymbol{D}_M, \boldsymbol{T}_M, \boldsymbol{B}_M$ — $M \times M$ matrices defined by

$$\boldsymbol{D}_M = \begin{pmatrix} \boldsymbol{I}_{\lceil M/2 \rceil} & \boldsymbol{O} \\ \boldsymbol{O} & -\boldsymbol{I}_{\lfloor M/2 \rfloor} \end{pmatrix} \tag{3}$$

$$\boldsymbol{T}_M = \begin{pmatrix} \boldsymbol{I}_{\lceil M/2 \rceil} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J}_{\lfloor M/2 \rfloor} \end{pmatrix} \tag{4}$$

and

$$\boldsymbol{B}_M = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} \boldsymbol{I}_{M/2} & \boldsymbol{I}_{M/2} \\ \boldsymbol{I}_{M/2} & -\boldsymbol{I}_{M/2} \end{pmatrix}, & M\text{: even} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} \boldsymbol{I}_{\lfloor M/2 \rfloor} & & \boldsymbol{I}_{\lfloor M/2 \rfloor} \\ & \sqrt{2} & \\ \boldsymbol{I}_{\lfloor M/2 \rfloor} & & -\boldsymbol{I}_{\lfloor M/2 \rfloor} \end{pmatrix}, & M\text{: odd} \end{cases} \tag{5}$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the integer value of x and the smallest integer greater than or equal to x, respectively.

$\mathbf{1}$ — $D \times 1$ vector whose elements are all '1.'

$\mathbf{1}^S$ — $D \times 1$ vector defined by $[\mathbf{1}^S]_k = 1$ for $k \in S$; otherwise 0, where $S \subseteq \{0, 1, \cdots, D-1\}$.

$\mathcal{N}$ — Set of $D \times 1$ integer vectors.

$\mathcal{N}(\boldsymbol{N})$ — Set of integer vectors in the fundamental parallelepiped generated with a $D \times D$ nonsingular integer matrix $\boldsymbol{N}$, which is defined by $\mathcal{N}(\boldsymbol{N}) = \{\boldsymbol{N}\boldsymbol{x} \in \mathcal{N} | \boldsymbol{x} \in [0,1)^D\}$, where $[a,b)^D$ denotes the set of $D \times 1$ vectors $\boldsymbol{x}$ so that the dth component satisfies $a \le x_d < b$ [1].

$J(\boldsymbol{N}) = |\det(\boldsymbol{N})|$ — absolute determinant of $\boldsymbol{N}$, which equals the number of elements in $\mathcal{N}(\boldsymbol{N})$.

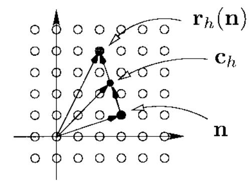


Fig. 1. Reflection with respect to $\boldsymbol{c}_h$ $(D = 2)$.

The product of an $M \times M$ square matrix sequence $\boldsymbol{A}_n$ is represented as

$$\prod_{n=N_S}^{N_E} \boldsymbol{A}_n = \boldsymbol{A}_{N_E} \boldsymbol{A}_{N_E-1} \cdots \boldsymbol{A}_{N_S}, \qquad N_S \le N_E. \tag{6}$$

In addition, the superscript 'T' on a vector or matrix denotes the transposition. Furthermore, the tilde notation '$\tilde{\ }$' over a vector or matrix denotes the paraconjugation [1], for example, $\tilde{\boldsymbol{E}}(\boldsymbol{z}) = \boldsymbol{E}_*^T(\boldsymbol{z}^{-1})$, where the subscript '$*$' denotes the complex conjugation of the coefficients.

II. Linear-Phase Property

As a preliminary, this section reviews the LP condition of MD filters and deals with the reflection invariance of their region of support.

A. MD-LP Filters

Let $H(\boldsymbol{z})$ be a D-dimensional filter. If $H(\boldsymbol{z})$ satisfies (7), it is said to be LP

$$H(\boldsymbol{z}) = \boldsymbol{z}^{-2\boldsymbol{c}_h} \gamma H(\boldsymbol{z}^{-\boldsymbol{I}}) \tag{7}$$

where $\boldsymbol{c}_h$ is a $D \times 1$ vector, which represents the center of filter $H(\boldsymbol{z})$, and $\boldsymbol{c}_h \in \frac{1}{2}\mathcal{N}$. γ is '1' when the filter $H(\boldsymbol{z})$ is symmetric and '-1' when it is antisymmetric with respect to (w.r.t.) the center $\boldsymbol{c}_h$.

In the following, we show a theorem with regard to the region of support.

Theorem 1: Let $\mathcal{N}_h$ be the region of support of a filter $H(\boldsymbol{z})$. If the filter $H(\boldsymbol{z})$ is LP, then

$$\{\boldsymbol{r}_h(\boldsymbol{n}) | \boldsymbol{n} \in \mathcal{N}_h\} = \mathcal{N}_h \tag{8}$$

holds, where $\boldsymbol{c}_h$ is the center, and $\boldsymbol{r}_h(\boldsymbol{n}) = 2\boldsymbol{c}_h - \boldsymbol{n}$ is the *reflection* w.r.t. $\boldsymbol{c}_h$ (see Fig. 1).

Proof: As the time-domain representation of (7), we have

$$h(\boldsymbol{n}) = \gamma h(\boldsymbol{r}_h(\boldsymbol{n})). \tag{9}$$

Hence, the relation $\mathcal{N}_h' = \{\boldsymbol{r}_h(\boldsymbol{n}) | \boldsymbol{n} \in \mathcal{N}_h\} \subseteq \mathcal{N}_h$ holds. In addition, it can be verified that $\{\boldsymbol{r}_h(\boldsymbol{r}_h(\boldsymbol{n})) | \boldsymbol{n} \in \mathcal{N}_h - \mathcal{N}_h'\} \subseteq$

$\mathcal{N}_h'$. The fact that $\boldsymbol{r}_h(\boldsymbol{r}_h(\boldsymbol{n})) = \boldsymbol{n}$ implies that $\mathcal{N}_h - \mathcal{N}_h' \subseteq \mathcal{N}_h'$. The only solution is $\mathcal{N}_h' = \mathcal{N}_h$. Therefore, (8) holds. ∎

The property expressed by (8) is referred to as *reflection invariance*, and such a region of support is said to be *reflection invariant*.

B. Polyphase Representation

Taking Theorem 1 into account, let us present the LP condition in the polyphase representation.

It is known that any MD filter $H(\boldsymbol{z})$ can be represented in terms of the polyphase filters with a nonsingular integer matrix $\boldsymbol{M}$, which is referred to as a *decimation matrix* or *factor*, as

$$H(\boldsymbol{z}) = \sum_{\boldsymbol{m} \in \mathcal{N}(\boldsymbol{M})} \boldsymbol{z}^{-\boldsymbol{m}} E_{\boldsymbol{m}}(\boldsymbol{z}^{\boldsymbol{M}}) \tag{10}$$

where $E_{\boldsymbol{m}}(\boldsymbol{z})$ denotes the $\boldsymbol{m}$th type-I polyphase filter of $H(\boldsymbol{z})$ w.r.t. the decimation matrix $\boldsymbol{M}$ [1].

Now, let us show three lemmas with regard to extension of the region of support $\mathcal{N}(\boldsymbol{M})$ and consider the LP property in the polyphase representation. In the following, we let $\boldsymbol{\Xi} = \mathrm{diag}(N_0+1, N_1+1, \cdots, N_{D-1}+1)$ with positive integers N_d and refer to $\boldsymbol{\Xi}$ as an *extension matrix*.

Lemma 2: Consider an extension matrix $\boldsymbol{\Xi}$. The region of support $\mathcal{N}(\boldsymbol{\Xi})$ is reflection invariant w.r.t. the center

$$\boldsymbol{c}_{\Xi} = \tfrac{1}{2}\overline{\boldsymbol{n}} \tag{11}$$

where $\overline{\boldsymbol{n}} = (N_0, N_1, \cdots, N_{D-1})^T$.

Proof: Since the region of support $\mathcal{N}(\boldsymbol{\Xi})$ forms a hypercube, the relation $\{\overline{\boldsymbol{n}} - \boldsymbol{n} | \boldsymbol{n} \in \mathcal{N}(\boldsymbol{\Xi})\} = \mathcal{N}(\boldsymbol{\Xi})$ holds. Hence, the reflection invariance is satisfied w.r.t. the center $\boldsymbol{c}_{\Xi}$ in (11).

Lemma 3: Consider the product of a decimation matrix $\boldsymbol{M}$ with an extension matrix $\boldsymbol{\Xi}$, that is, $\boldsymbol{M}\boldsymbol{\Xi}$. The region of support $\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi})$ is expressed as

$$\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi}) = \{\boldsymbol{M}\boldsymbol{i} + \boldsymbol{m} | \boldsymbol{i} \in \mathcal{N}(\boldsymbol{\Xi}), \boldsymbol{m} \in \mathcal{N}(\boldsymbol{M})\}. \tag{12}$$

Proof: From the fact that $\{\boldsymbol{\Xi}\boldsymbol{x} | \boldsymbol{x} \in [0,1)^D\} = \{\boldsymbol{i} + \boldsymbol{x} | \boldsymbol{i} \in \mathcal{N}(\boldsymbol{\Xi}), \boldsymbol{x} \in [0,1)^D\}$

$$\begin{aligned}\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi}) &= \{\boldsymbol{M}\boldsymbol{\Xi}\boldsymbol{x}) \in \mathcal{N} | \boldsymbol{x} \in [0,1)^D\} \\ &= \{\boldsymbol{M}(\boldsymbol{i} + \boldsymbol{x}) \in \mathcal{N} | \boldsymbol{i} \in \mathcal{N}(\boldsymbol{\Xi}), \boldsymbol{x} \in [0,1)^D\} \\ &= \{\boldsymbol{M}\boldsymbol{i} + \boldsymbol{M}\boldsymbol{x} \in \mathcal{N} | \boldsymbol{i} \in \mathcal{N}(\boldsymbol{\Xi}), \boldsymbol{x} \in [0,1)^D\}.\end{aligned} \tag{13}$$

Since $\{\boldsymbol{M}\boldsymbol{x} \in \mathcal{N} | \boldsymbol{x} \in [0,1)^D\} = \mathcal{N}(\boldsymbol{M})$ and $\boldsymbol{M}\boldsymbol{i} \in \mathcal{N}$, (12) is obtained.

Lemma 4: If and only if $\mathcal{N}(\boldsymbol{M})$ is reflection invariant w.r.t. some center $\boldsymbol{c}_M$, the extended region of support $\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi})$ is also reflection invariant w.r.t. the center $\boldsymbol{c}_h$

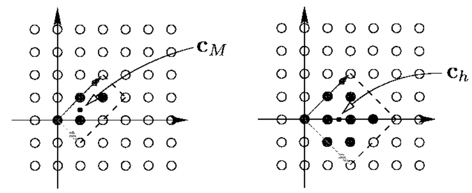
$$\boldsymbol{c}_h = \boldsymbol{M}\boldsymbol{c}_{\Xi} + \boldsymbol{c}_M \tag{14}$$

where $\boldsymbol{c}_{\Xi}$ is the center of $\mathcal{N}(\boldsymbol{\Xi})$.

Proof: From the assumption and Lemma 2, $\mathcal{N}(\boldsymbol{M})$ and $\mathcal{N}(\boldsymbol{\Xi})$ are reflection invariant. Hence

$$\mathcal{N}(\boldsymbol{M}) = \{\boldsymbol{r}_M(\boldsymbol{m}) | \boldsymbol{m} \in \mathcal{N}(\boldsymbol{M})\} \tag{15}$$

$$\mathcal{N}(\boldsymbol{\Xi}) = \{\boldsymbol{r}_{\Xi}(\boldsymbol{i}) | \boldsymbol{i} \in \mathcal{N}(\boldsymbol{\Xi})\} \tag{16}$$



(a) (b)

Fig. 2. Example of extension $(D = 2)$. (a) and (b) show $\mathcal{N}(\boldsymbol{M})$ and $\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi})$, respectively, where $\boldsymbol{M} = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$, $\boldsymbol{\Xi} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, and $\overline{\boldsymbol{n}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $\boldsymbol{c}_M$ and $\boldsymbol{c}_h$ are the centers of $\mathcal{N}(\boldsymbol{M})$ and $\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi})$, respectively.

are satisfied, where $\boldsymbol{r}_M(\boldsymbol{m}) = 2\boldsymbol{c}_M - \boldsymbol{m}$ and $\boldsymbol{r}_{\Xi}(\boldsymbol{i}) = 2\boldsymbol{c}_{\Xi} - \boldsymbol{i}$. The above relations and Lemma 3 then lead to

$$\begin{aligned}\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi}) &= \{\boldsymbol{M}\boldsymbol{r}_{\Xi}(\boldsymbol{i}) + \boldsymbol{r}_M(\boldsymbol{m}) | \boldsymbol{i} \in \mathcal{N}(\boldsymbol{\Xi}), \boldsymbol{m} \in \mathcal{N}(\boldsymbol{M})\} \\ &= \{2(\boldsymbol{M}\boldsymbol{c}_{\Xi} + \boldsymbol{c}_M) - (\boldsymbol{M}\boldsymbol{i} + \boldsymbol{m}) | \boldsymbol{i} \in \mathcal{N}(\boldsymbol{\Xi}) \\ &\qquad \boldsymbol{m} \in \mathcal{N}(\boldsymbol{M})\} \\ &= \{\boldsymbol{r}_h(\boldsymbol{k}) | \boldsymbol{k} \in \mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi})\}\end{aligned} \tag{17}$$

where $\boldsymbol{r}_h(\boldsymbol{k}) = 2(\boldsymbol{M}\boldsymbol{c}_{\Xi} + \boldsymbol{c}_M) - \boldsymbol{k}$. That is, $\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi})$ is reflection invariant w.r.t. $\boldsymbol{M}\boldsymbol{c}_{\Xi} + \boldsymbol{c}_M$. Conversely, if $\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi})$ is reflection invariant w.r.t. some center $\boldsymbol{c}_h$, then

$$\begin{aligned}\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi}) &= \{2\boldsymbol{c}_h - \boldsymbol{k} | \boldsymbol{k} \in \mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi})\} \\ &= \{2\boldsymbol{c}_h - (\boldsymbol{M}\boldsymbol{i} + \boldsymbol{m}) | \boldsymbol{i} \in \mathcal{N}(\boldsymbol{\Xi}), \boldsymbol{m} \in \mathcal{N}(\boldsymbol{M})\} \\ &= \{\boldsymbol{M}\boldsymbol{r}_{\Xi}(\boldsymbol{i}) + (2(\boldsymbol{c}_h - \boldsymbol{M}\boldsymbol{c}_{\Xi}) - \boldsymbol{m}) | \boldsymbol{i} \in \mathcal{N}(\boldsymbol{\Xi}) \\ &\qquad \boldsymbol{m} \in \mathcal{N}(\boldsymbol{M})\} \\ &= \{\boldsymbol{M}\boldsymbol{i} + (2(\boldsymbol{c}_h - \boldsymbol{M}\boldsymbol{c}_{\Xi}) - \boldsymbol{m}) | \boldsymbol{i} \in \mathcal{N}(\boldsymbol{\Xi}) \\ &\qquad \boldsymbol{m} \in \mathcal{N}(\boldsymbol{M})\}.\end{aligned} \tag{18}$$

Comparing this result with (12), it can be proven that $\mathcal{N}(\boldsymbol{M})$ is reflection invariant w.r.t. $\boldsymbol{c}_M = \boldsymbol{c}_h - \boldsymbol{M}\boldsymbol{c}_{\Xi}$.

Lemma 4 guarantees that the region of support $\mathcal{N}(\boldsymbol{M})$ can be extended by the matrix $\boldsymbol{\Xi}$ while holding the reflection invariance. Fig. 2 shows an example of the extension.

On the basis of these lemmas, let us show the LP condition in the polyphase representation.

Theorem 5: Consider a filter $H(\boldsymbol{z})$ with an extended region of support $\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi})$, and let $E_{\boldsymbol{m}}(\boldsymbol{z})$ be the $\boldsymbol{m}$th type-I polyphase filter w.r.t. $\boldsymbol{M}$. If and only if $H(\boldsymbol{z})$ is LP with some center $\boldsymbol{c}_h$, then

$$E_{\boldsymbol{m}}(\boldsymbol{z}) = \boldsymbol{z}^{-2\boldsymbol{c}_{\Xi}} \gamma E_{\boldsymbol{r}_M(\boldsymbol{m})}(\boldsymbol{z}^{-\boldsymbol{I}}), \qquad \boldsymbol{m} \in \mathcal{N}(\boldsymbol{M}) \tag{19}$$

holds, where $\boldsymbol{c}_{\Xi}$ is the center of $\mathcal{N}(\boldsymbol{\Xi})$ and $\boldsymbol{r}_M(\boldsymbol{m}) = 2\boldsymbol{c}_M - \boldsymbol{m}$, and $\boldsymbol{c}_M = \boldsymbol{c}_h - \boldsymbol{M}\boldsymbol{c}_{\Xi}$. Additionally, γ is '1' for a symmetric filter or '-1' for an antisymmetric filter.

Proof: From Lemma 4, $\mathcal{N}(\boldsymbol{M})$ is reflection invariant w.r.t. the center $\boldsymbol{c}_M$. Hence, (7) is expressed as

$$\begin{aligned}H(\boldsymbol{z}) &= \boldsymbol{z}^{-2\boldsymbol{c}_h} \gamma H(\boldsymbol{z}^{-\boldsymbol{I}}) \\ &= \boldsymbol{z}^{-2(\boldsymbol{M}\boldsymbol{c}_{\Xi} + \boldsymbol{c}_M)} \gamma \sum_{\boldsymbol{m} \in \mathcal{N}(\boldsymbol{M})} \boldsymbol{z}^{\boldsymbol{m}} E_{\boldsymbol{m}}(\boldsymbol{z}^{-\boldsymbol{M}}) \\ &= \boldsymbol{z}^{-2\boldsymbol{M}\boldsymbol{c}_{\Xi}} \gamma \sum_{\boldsymbol{m} \in \mathcal{N}(\boldsymbol{M})} \boldsymbol{z}^{-\boldsymbol{r}_M(\boldsymbol{m})} E_{\boldsymbol{m}}(\boldsymbol{z}^{-\boldsymbol{M}}) \\ &= \sum_{\boldsymbol{m} \in \mathcal{N}(\boldsymbol{M})} \boldsymbol{z}^{-\boldsymbol{m}} \{\boldsymbol{z}^{-2\boldsymbol{M}\boldsymbol{c}_{\Xi}} \gamma E_{\boldsymbol{r}_M(\boldsymbol{m})}(\boldsymbol{z}^{-\boldsymbol{M}})\}.\end{aligned} \tag{20}$$

Compared with (10), (20) leads to (19).

Note that the vector $2\boldsymbol{c}_\Xi$ consists of the order of the polyphase matrix $\boldsymbol{E}(\boldsymbol{z})$, that is, $2\boldsymbol{c}_\Xi = \overline{\boldsymbol{n}} = (N_0, N_1, \cdots, N_{D-1})^T$.

C. Ordering of $E_{\boldsymbol{m}}(\boldsymbol{z})$

Now, for the sake of convenience, let us order the polyphase filters $E_{\boldsymbol{m}}(\boldsymbol{z})$ and modify their indexes as

$$E_\ell(\boldsymbol{z}) = E_{\boldsymbol{m}_\ell}(\boldsymbol{z}), \qquad \ell = 0, 1, 2, \cdots, M-1 \tag{21}$$

where $\boldsymbol{m}_\ell \in \mathcal{N}(\boldsymbol{M})$, and $M = J(\boldsymbol{M})$. By using this notation, let us define an $M \times 1$ polyphase vector $\boldsymbol{e}(\boldsymbol{z})$ by

$$\boldsymbol{e}(\boldsymbol{z}) = (E_0(\boldsymbol{z}) \quad E_1(\boldsymbol{z}) \quad \cdots \quad E_{M-1}(\boldsymbol{z}))^T. \tag{22}$$

This vector $\boldsymbol{e}(\boldsymbol{z})$ is related to $H(\boldsymbol{z})$ as

$$H(\boldsymbol{z}) = \boldsymbol{e}^T(\boldsymbol{z}^{\boldsymbol{M}})\boldsymbol{d}_{\boldsymbol{M}}(\boldsymbol{z}) \tag{23}$$

where

$$\boldsymbol{d}_{\boldsymbol{M}}(\boldsymbol{z}) = (\boldsymbol{z}^{-\boldsymbol{m}_0} \quad \boldsymbol{z}^{-\boldsymbol{m}_1} \quad \cdots \quad \boldsymbol{z}^{-\boldsymbol{m}_{M-1}})^T$$
$$\boldsymbol{m}_\ell \in \mathcal{N}(\boldsymbol{M}). \tag{24}$$

On the assumption that $\mathcal{N}(\boldsymbol{M})$ is reflection invariant, the polyphase filters can always be ordered to satisfy

$$\boldsymbol{r}_M(\boldsymbol{m}_\ell) = \boldsymbol{m}_{M-1-\ell}, \qquad \ell = 0, 1, 2, \cdots, M-1. \tag{25}$$

If the elements are ordered according to the above rule, then the LP condition in (19) simplifies to

$$\boldsymbol{e}^T(\boldsymbol{z}) = \boldsymbol{z}^{-2\boldsymbol{c}_\Xi}\gamma\boldsymbol{e}^T(\boldsymbol{z}^{-\boldsymbol{I}})\boldsymbol{J}_M. \tag{26}$$

III. MD-LPPU Filter Banks

In this section, we review multidimensional (MD) maximally decimated filter banks and discuss the paraunitary (PU) and linear-phase (LP) properties. The class dealt with in this paper is also clarified.

A. Review of MD Filter Banks

Fig. 3(a) shows a parallel structure of MD maximally decimated filter banks with a factor $\boldsymbol{M}$, where $\downarrow\boldsymbol{M}$ and $\uparrow\boldsymbol{M}$ denote the down- and upsamplers with the factor $\boldsymbol{M}$, respectively. The number of channels is $M = J(\boldsymbol{M})$. If $\boldsymbol{M}$ is a diagonal matrix, then we refer to such systems as *rectangular decimation filter banks*; otherwise, they are *nonrectangular decimation filter banks*.

Decomposing each filter into the polyphase filters, the parallel structure can be equivalently represented as shown in Fig. 3(b), where $\boldsymbol{E}(\boldsymbol{z})$ is the type-I polyphase matrix of analysis bank, and $\boldsymbol{R}(\boldsymbol{z})$ is the type-II polyphase matrix of synthesis bank [1]. These polyphase matrices are related to $H_k(\boldsymbol{z})$ and $F_k(\boldsymbol{z})$ as

$$\boldsymbol{h}(\boldsymbol{z}) = \begin{pmatrix} H_0(\boldsymbol{z}) \\ H_1(\boldsymbol{z}) \\ \vdots \\ H_{M-1}(\boldsymbol{z}) \end{pmatrix} = \boldsymbol{E}(\boldsymbol{z}^{\boldsymbol{M}})\boldsymbol{d}_{\boldsymbol{M}}(\boldsymbol{z}) \tag{27}$$

$$\boldsymbol{f}(\boldsymbol{z}) = (F_0(\boldsymbol{z}) \quad F_1(\boldsymbol{z}) \quad \cdots \quad F_{M-1}(\boldsymbol{z}))^T$$
$$= \boldsymbol{d}_{\boldsymbol{M}}^T(\boldsymbol{z}^{-\boldsymbol{I}})\boldsymbol{R}(\boldsymbol{z}^{\boldsymbol{M}}). \tag{28}$$

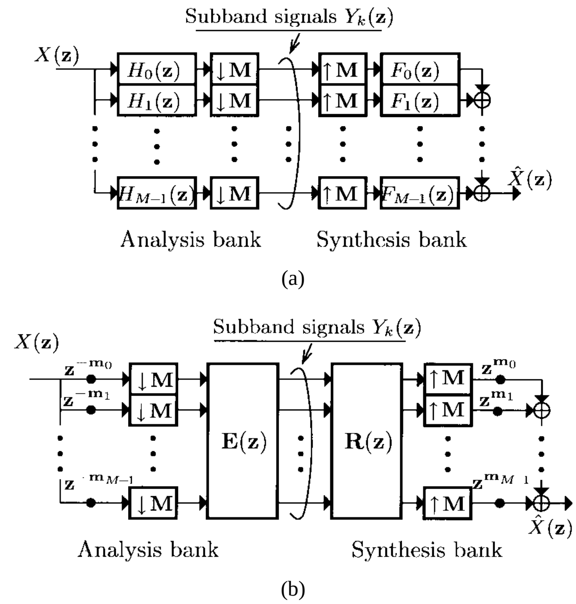


Fig. 3. Structures of MD filter banks. (a) Parallel structure. (b) Polyphase structure.

B. Class of Filter Banks

In order to clarify which class of filter banks is dealt with in this paper, we show the properties that our proposed filter banks possess.

1) Paraunitary (PU) Property: Let us construct MD filter banks to be *paraunitary* (PU) [1]. If the polyphase matrix $\boldsymbol{E}(\boldsymbol{z})$ satisfies (29), it is said to be paraunitary.

$$\tilde{\boldsymbol{E}}(\boldsymbol{z})\boldsymbol{E}(\boldsymbol{z}) = \boldsymbol{I}_M. \tag{29}$$

This condition is sufficient to construct PR filter banks because the simple choice of the synthesis bank as $\boldsymbol{R}(\boldsymbol{z}) = \boldsymbol{z}^{-\boldsymbol{n}}\tilde{\boldsymbol{E}}(\boldsymbol{z})$ with some $D \times 1$ integer vector $\boldsymbol{n}$ results in $\boldsymbol{R}(\boldsymbol{z})\boldsymbol{E}(\boldsymbol{z}) = \boldsymbol{z}^{-\boldsymbol{n}}\boldsymbol{I}_M$, which shows the system is PR [1]. Thus, in the following, we deal only with an analysis bank. Since we let filters have real coefficients, we actually consider those holding the property $\boldsymbol{E}^T(\boldsymbol{z}^{-\boldsymbol{I}})\boldsymbol{E}(\boldsymbol{z}) = \boldsymbol{I}_M$.

2) Linear-Phase (LP) Property: The individual filters in our proposed filter banks are designed to be LP. In order to guarantee this property, we choose the factor $\boldsymbol{M}$ such that $\mathcal{N}(\boldsymbol{M})$ is reflection invariant w.r.t. some center $\boldsymbol{c}_M$, and let the region of support of the filters be $\mathcal{N}(\boldsymbol{M}\boldsymbol{\Xi})$ according to Theorem 5 by using an extension matrix $\boldsymbol{\Xi}$.

Since the LP property of each filter $H_k(\boldsymbol{z})$ can be expressed as in (26) in terms of its polyphase vector, the LP property of analysis bank $\boldsymbol{E}(\boldsymbol{z})$ can be represented as

$$\boldsymbol{E}(\boldsymbol{z}) = \boldsymbol{z}^{-2\boldsymbol{c}_\Xi}\boldsymbol{D}_M\boldsymbol{E}(\boldsymbol{z}^{-\boldsymbol{I}})\boldsymbol{J}_M \tag{30}$$

where we assume that $H_k(\boldsymbol{z})$ for $k = 0, 1, \cdots, \lceil M/2 \rceil - 1$ are symmetric and that the rest are antisymmetric.

Here, note that the polyphase components are ordered according to (25), and the number of symmetric and antisymmetric filters are determined on the basis of Theorem 6, which is proven in the same way as Theorem 1 shown in [6].

Theorem 6: Consider matrix-$\boldsymbol{M}$ LP PR filter banks, whose filters all have the extended region of support $\mathcal{N}(\boldsymbol{M\Xi})$.

1) If $M = J(\boldsymbol{M})$ is even, there are $M/2$ symmetric and $M/2$ antisymmetric filters.
2) If $M = J(\boldsymbol{M})$ is odd, there are $(M+1)/2$ symmetric and $(M-1)/2$ antisymmetric filters.

Proof: Primarily, the LP condition is represented as $\boldsymbol{E}(\boldsymbol{z}) = \boldsymbol{z}^{-2\boldsymbol{c}_\Xi}\boldsymbol{\Gamma}_M\boldsymbol{E}(\boldsymbol{z}^{-\boldsymbol{I}})\boldsymbol{J}_M$ instead of (30), where $\boldsymbol{\Gamma}_M$ is an arbitrary $M \times M$ diagonal matrix whose diagonal elements are '1' or '−1.'

By taking the trace of both sides of (30) and using the fact that $\boldsymbol{E}(\boldsymbol{z})$ is invertible, we have

$$\begin{aligned}\mathrm{tr}(\boldsymbol{\Gamma}_M) &= \mathrm{tr}(\boldsymbol{z}^{2\boldsymbol{c}_\Xi}\boldsymbol{E}(\boldsymbol{z})\boldsymbol{J}_M\boldsymbol{E}^{-1}(\boldsymbol{z}^{-\boldsymbol{I}}))\\ &= \mathrm{tr}(\boldsymbol{z}^{2\boldsymbol{c}_\Xi}\boldsymbol{E}^{-1}(\boldsymbol{z}^{-\boldsymbol{I}})\boldsymbol{E}(\boldsymbol{z})\boldsymbol{J}_M)\end{aligned} \tag{31}$$

where $\mathrm{tr}(\boldsymbol{N})$ is the trace of $\boldsymbol{N}$. This equation should be satisfied for any value of $\boldsymbol{z}$. Let us substitute $\boldsymbol{z} = \boldsymbol{1}$ into this.

$$\begin{aligned}\mathrm{tr}(\boldsymbol{\Gamma}_M) &= \mathrm{tr}(\boldsymbol{E}^{-1}(\boldsymbol{1})\boldsymbol{E}(\boldsymbol{1})\boldsymbol{J}_M)\\ &= \mathrm{tr}(\boldsymbol{J}_M).\end{aligned} \tag{32}$$

The last equation implies that $\mathrm{tr}(\boldsymbol{\Gamma}_M) = 0$ for even M and $\mathrm{tr}(\boldsymbol{\Gamma}_M) = 1$ for odd M. In other words, if M is even, then the number of symmetric filters is equal to the number of antisymmetric ones, and if M is odd, then there is one extra symmetric filter.

3) Causality: Let $\boldsymbol{E}(\boldsymbol{z})$ be causal in all dimensions since many results on 1-D LPPUFB's are derived under the condition that polyphase matrices are causal. Under this assumption, most of the results can be applied to MD ones. Note that this does not necessarily mean the causality of $\boldsymbol{h}(\boldsymbol{z})$, which depends on the choice of the factor $\boldsymbol{M}$.

For example, let us consider the 2-D case that the factor is given by $\boldsymbol{M} = \begin{pmatrix}2 & 1\\ 2 & -1\end{pmatrix}$, and the polyphase matrix is provided as

$$\boldsymbol{E}(\boldsymbol{z}) = \begin{pmatrix}1 & 0 & 0 & \boldsymbol{z}^{-\boldsymbol{1}^{\{0\}}}\\ 0 & 1 & \boldsymbol{z}^{-\boldsymbol{1}^{\{0\}}} & 0\\ 1 & 0 & 0 & -\boldsymbol{z}^{-\boldsymbol{1}^{\{0\}}}\\ 0 & 1 & -\boldsymbol{z}^{-\boldsymbol{1}^{\{0\}}} & 0\end{pmatrix} \tag{33}$$

where $\boldsymbol{z}^{-\boldsymbol{1}^{\{0\}}} = z_0^{-1}$. The above matrix is easily found to be causal in all dimensions. From (27), we it can observe that the corresponding analysis filters can be expressed as

$$\begin{aligned}\boldsymbol{h}(\boldsymbol{z}) &= \begin{pmatrix}1 & 0 & 0 & \boldsymbol{z}^{-\boldsymbol{M}1^{\{0\}}}\\ 0 & 1 & \boldsymbol{z}^{-\boldsymbol{M}1^{\{0\}}} & 0\\ 1 & 0 & 0 & -\boldsymbol{z}^{-\boldsymbol{M}1^{\{0\}}}\\ 0 & 1 & -\boldsymbol{z}^{-\boldsymbol{M}1^{\{0\}}} & 0\end{pmatrix}\begin{pmatrix}1\\ z_0^{-1}z_1^{-1}\\ z_0^{-1}\\ z_0^{-2}z_1^{-1}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0 & 0 & z_0^{-1}z_1\\ 0 & 1 & z_0^{-1}z_1 & 0\\ 1 & 0 & 0 & -z_0^{-1}z_1\\ 0 & 1 & -z_0^{-1}z_1 & 0\end{pmatrix}\begin{pmatrix}1\\ z_0^{-1}z_1^{-1}\\ z_0^{-1}\\ z_0^{-2}z_1^{-1}\end{pmatrix}\\ &= \begin{pmatrix}1 + z_0^{-3}\\ z_0^{-1}z_1^{-1} + z_0^{-2}z_1\\ z_0^{-1}z_1^{-1} - z_0^{-2}z_1\\ 1 - z_0^{-3}\end{pmatrix}.\end{aligned} \tag{34}$$

This last equation illustrates that the analysis bank is not causal because the advance operator z_1 is involved, but this is permissible under our proposed structure.

IV. Lattice Structure

In the previous section, we discussed the class of filter banks dealt with in this paper. This section provides our proposed lattice structure for such filter banks. In the following, we first provide the structure for an even M and then that for an odd M. Their minimality and the no-DC-leakage condition are also shown.

A. For Even M

To construct a lattice structure of MD-LPPUFB's, we consider formulating the order-increasing process of the polyphase matrix $\boldsymbol{E}(\boldsymbol{z})$ while keeping both of the LP and PU properties. This approach is motivated from the process for that of 1-D LPPUFB's [3], [6]–[8].

Let $\boldsymbol{E}_m(\boldsymbol{z})$ be a polyphase matrix, whose dth-dimension order is m. We consider increasing the dth-dimension order m to $m+1$ as

$$\boldsymbol{E}_{m+1}(\boldsymbol{z}) = \boldsymbol{R}_{m+1}^{\{d\}}\boldsymbol{Q}^{\{d\}}(\boldsymbol{z})\boldsymbol{E}_m(\boldsymbol{z}) \tag{35}$$

where

$$\boldsymbol{R}_n^{\{d\}} = \begin{pmatrix}\boldsymbol{W}_n^{\{d\}} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{U}_n^{\{d\}}\end{pmatrix} \tag{36}$$

$$\boldsymbol{Q}^{\{d\}}(\boldsymbol{z}) = \boldsymbol{B}_M\boldsymbol{\Lambda}^{\{d\}}(\boldsymbol{z})\boldsymbol{B}_M. \tag{37}$$

$\boldsymbol{W}_n^{\{d\}}$, and $\boldsymbol{U}_n^{\{d\}}$ denote $M/2 \times M/2$ orthonormal matrices, and in addition

$$\boldsymbol{\Lambda}^{\{d\}}(\boldsymbol{z}) = \begin{pmatrix}\boldsymbol{I}_{M/2} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{z}^{-\boldsymbol{1}^{\{d\}}}\boldsymbol{I}_{M/2}\end{pmatrix}. \tag{38}$$

Although $\boldsymbol{z}^{-\boldsymbol{1}^{\{d\}}}$ can simply be represented as z_d^{-1}, we still use the vector notation for the consistent expression in multidimensions.

It can easily be verified that the PU property of $\boldsymbol{E}_m(\boldsymbol{z})$ as in (29) results in that of $\boldsymbol{E}_{m+1}(\boldsymbol{z})$ since $\boldsymbol{R}_n^{\{d\}}$ and $\boldsymbol{Q}^{\{d\}}(\boldsymbol{z})$ are PU. In addition, the LP property of $\boldsymbol{E}_m(\boldsymbol{z})$, as in (30), provides that of $\boldsymbol{E}_{m+1}(\boldsymbol{z})$. Let us verify this fact.

Equation (35) can be rewritten as

$$\boldsymbol{E}_m(\boldsymbol{z}) = \boldsymbol{Q}^{\{d\}}(\boldsymbol{z}^{-\boldsymbol{I}})\boldsymbol{R}_{m+1}^{\{d\}T}\boldsymbol{E}_{m+1}(\boldsymbol{z}). \tag{39}$$

By substituting the above equation into the LP condition of $\boldsymbol{E}_m(\boldsymbol{z})$, that is

$$\boldsymbol{E}_m(\boldsymbol{z}) = \boldsymbol{z}^{-2\boldsymbol{c}_{\Xi m}}\boldsymbol{D}_M\boldsymbol{E}_m(\boldsymbol{z}^{-\boldsymbol{I}})\boldsymbol{J}_M \tag{40}$$

we have

$$\begin{aligned}&\boldsymbol{Q}^{\{d\}}(\boldsymbol{z}^{-\boldsymbol{I}})\boldsymbol{R}_{m+1}^{\{d\}T}\boldsymbol{E}_{m+1}(\boldsymbol{z})\\ &\quad= \boldsymbol{z}^{-2\boldsymbol{c}_{\Xi m}}\boldsymbol{D}_M\boldsymbol{Q}^{\{d\}}(\boldsymbol{z})\boldsymbol{R}_{m+1}^{\{d\}T}\boldsymbol{E}_{m+1}(\boldsymbol{z}^{-\boldsymbol{I}})\boldsymbol{J}_M\end{aligned} \tag{41}$$

where $\boldsymbol{c}_{\Xi m}$ is the center of $\mathcal{N}(\Xi_m)$, and Ξ_m is an extension matrix whose dth diagonal element is $m+1$, which is the dth dimension order $[2\boldsymbol{c}_{\Xi_m}]_d = m$ plus one.

From the fact that

$$\boldsymbol{R}_{m+1}^{\{d\}}\boldsymbol{Q}^{\{d\}}(\boldsymbol{z})\boldsymbol{D}_M\boldsymbol{Q}^{\{d\}}(\boldsymbol{z})\boldsymbol{R}_{m+1}^{\{d\}T} = \boldsymbol{z}^{-\boldsymbol{1}^{\{d\}}}\boldsymbol{D}_M \tag{42}$$

(41) is reduced to

$$\boldsymbol{E}_{m+1}(\boldsymbol{z}) = \boldsymbol{z}^{-2\boldsymbol{c}_{\Xi m+1}}\boldsymbol{D}_M\boldsymbol{E}_{m+1}(\boldsymbol{z}^{-\boldsymbol{I}})\boldsymbol{J}_M \tag{43}$$

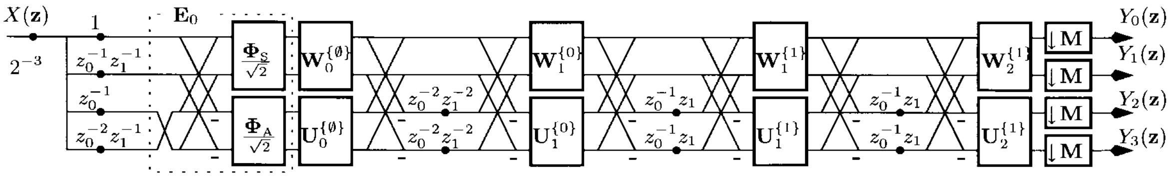


Fig. 4. Example of proposed lattice structure of MD-LPPUFB's, where $\boldsymbol{M} = \begin{pmatrix}2 & 1\\ 2 & -1\end{pmatrix}$, $\boldsymbol{\Xi} = \begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$, and $\overline{\boldsymbol{n}} = \begin{pmatrix}1\\ 2\end{pmatrix}$.

where $2\boldsymbol{c}_{\Xi m+1} = 2\boldsymbol{c}_{\Xi m} + \mathbf{1}^{\{d\}}$, namely, the dth-dimension order $[2\boldsymbol{c}_{\Xi_{m+1}}]_d$ equals $m+1$.

The last result implies that $\boldsymbol{E}_{m+1}(\boldsymbol{z})$ sufficiently satisfies the LP condition as in (30), and the order of $\boldsymbol{E}_{m+1}(\boldsymbol{z})$, e.g., $\overline{\boldsymbol{n}}_{m+1} = 2\boldsymbol{c}_{\Xi m+1}$, has one more dth-dimension order than that of $\boldsymbol{E}_m(\boldsymbol{z})$, e.g., $\overline{\boldsymbol{n}}_m = 2\boldsymbol{c}_{\Xi m}$, and the same order as each other for the other dimensions.

Therefore, by repeating the order-increasing process according to (35), we can extend the region of support of all filters while keeping the LP and PU properties. This process is applicable to any dimension. As a result, it can be verified that the following product form generates a lattice structure of a D-dimensional LPPUFB for even M:

$$\boldsymbol{E}(\boldsymbol{z}) = \left\{\prod_{d=0}^{D-1} \prod_{\substack{n=1\\ N_d \neq 0}}^{N_d} \boldsymbol{R}_n^{\{d\}} \boldsymbol{Q}^{\{d\}}(\boldsymbol{z})\right\} \boldsymbol{R}_0^{\{\emptyset\}} \boldsymbol{E}_0 \tag{44}$$

$$\boldsymbol{E}_0 = \begin{pmatrix}\boldsymbol{\Phi}_{\mathrm{S}} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{\Phi}_{\mathrm{A}}\end{pmatrix} \boldsymbol{B}_M \boldsymbol{T}_M \tag{45}$$

where $\boldsymbol{\Phi}_{\mathrm{S}}$ and $\boldsymbol{\Phi}_{\mathrm{A}}$ are $M/2 \times M/2$ orthonormal matrices, which are fixed during the design phase and contribute only to the starting guess. The superscript '$\{\emptyset\}$' on $\boldsymbol{R}_0$ has no special meaning except that it provides a consistent expression with the definition of $\boldsymbol{R}_n^{\{d\}}$. The matrix $\boldsymbol{E}_0$ is also PU and LP. In the case of $D=1$, (44) is reduced to the factorization of an even number of channel 1-D LPPUFB's whose filters all have a multiple of the number of channels M [3], [6]–[8]. Fig. 4 shows an example of our proposed lattice structure, where the relation as in (27) is used.

By controlling the matrices $\boldsymbol{W}_n^{\{d\}}$ and $\boldsymbol{U}_n^{\{d\}}$, we can design MD-LPPUFB's with the guarantee of the PU and LP properties. Since an $N \times N$ orthonormal matrix can be characterized by a combination of $N(N-1)/2$ Givens rotations with N sign parameters [1]–[3], designing such systems is made possible by means of an unconstrained nonlinear optimization process to minimize (or maximize) some object function by predetermining the sign parameters. We will give some design examples of our proposed structure in Section V.

B. For Odd M

For odd M, we first have to show a necessary condition for the components in the extension matrix $\boldsymbol{\Xi}$. The condition can be regarded as an extension of [6, Cor. 1] and [16, Th. 2] to the MD case.

Theorem 7: Consider matrix-$\boldsymbol{M}$ LP PR filter banks, whose filters all have the extended region of support $\mathcal{N}(\boldsymbol{M\Xi})$, where $\boldsymbol{\Xi} = \mathrm{diag}(N_0+1, N_1+1, \cdots, N_{D-1}+1)$, and $N_d \geq 1$. If $M = J(\boldsymbol{M})$ is odd, all of N_d for $d \in \{0, 1, \cdots, D-1\}$ are even.

Proof: Taking the determinant of both sides of (30), we have

$$(\boldsymbol{z}^{-2\boldsymbol{c}_\Xi})^M \det(\boldsymbol{D}_M \boldsymbol{E}(\boldsymbol{z}^{-\boldsymbol{I}}) \boldsymbol{J}_M) = \det(\boldsymbol{E}(\boldsymbol{z})). \tag{46}$$

Note that this equation has to be satisfied with any value of $\boldsymbol{z}$.

Let us define $\overline{\mathbf{1}}^{\{d\}} = \mathbf{1}^{\{\overline{d}\}} - \mathbf{1}^{\{d\}}$, where $d \in \{0, 1, \cdots, D-1\}$ and $\{\overline{d}\}$ is the complement set of $\{d\}$. For example, when $D=4$, $\overline{\mathbf{1}}^{\{1\}} = (1, -1, 1, 1)^T$.

For odd M, by substituting $\boldsymbol{z} = \overline{\mathbf{1}}^{\{d\}}$ into (46), we obtain

$$(-1)^{-N_d M} \det(\boldsymbol{D}_M)(-1)^{(M-1)/2} = 1 \tag{47}$$

where we use the multiplicative property of determinants and the facts that $2\boldsymbol{c}_\Xi = (N_0, N_1, \cdots, N_{D-1})^T$, $(\overline{\mathbf{1}}^{\{d\}})^{-\boldsymbol{I}} = \overline{\mathbf{1}}^{\{d\}}$, $\boldsymbol{E}(\boldsymbol{z})$ is invertible, and $\det(\boldsymbol{J}_M) = (-1)^{M(M-1)/2} = (-1)^{(M-1)/2}$ for odd M [16].

When $(M-1)/2$ is even, $\det(\boldsymbol{D}_M) = 1$ and $(-1)^{(M-1)/2} = 1$; otherwise, $\det(\boldsymbol{D}_M) = -1$ and $(-1)^{(M-1)/2} = -1$. Therefore, $(-1)^{-N_d M}$ has to be 1. In other words, since M is odd, N_d has to be even for any d. ■

When the number of channels M is odd, it can be verified that the product form in (48) generates a lattice structure of MD-LPPUFB's. Now, we redefine $\boldsymbol{\Phi}_{\mathrm{S}}$ and $\boldsymbol{\Phi}_{\mathrm{A}}$ in (45) as $\lceil M/2 \rceil \times \lceil M/2 \rceil$ and $\lfloor M/2 \rfloor \times \lfloor M/2 \rfloor$ orthonormal matrices, respectively, so that the expression of (45) can be used in common.

$$\boldsymbol{E}(\boldsymbol{z}) = \left\{\prod_{d=0}^{D-1} \prod_{\substack{\ell=1\\ L_d \neq 0}}^{L-d} \boldsymbol{R}_{\mathrm{E}\ell}^{\{d\}} \boldsymbol{Q}_{\mathrm{E}}^{\{d\}}(\boldsymbol{z}) \boldsymbol{R}_{\mathrm{O}\ell}^{\{d\}} \boldsymbol{Q}_{\mathrm{O}}^{\{d\}}(\boldsymbol{z})\right\} \boldsymbol{R}_{\mathrm{E}0}^{\{\emptyset\}} \boldsymbol{E}_0 \tag{48}$$

where $L_d = N_d/2$, which is guaranteed to be an integer since N_d is even. The matrices $\boldsymbol{R}_{\mathrm{E}\ell}^{\{d\}}$, $\boldsymbol{R}_{\mathrm{O}\ell}^{\{d\}}$, $\boldsymbol{Q}_{\mathrm{E}}^{\{d\}}(\boldsymbol{z})$, and $\boldsymbol{Q}_{\mathrm{O}}^{\{d\}}(\boldsymbol{z})$ are

$$\boldsymbol{R}_{\mathrm{E}\ell}^{\{d\}} = \begin{pmatrix}\boldsymbol{W}_{\mathrm{E}\ell}^{\{d\}} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{U}_{\mathrm{E}\ell}^{\{d\}}\end{pmatrix} \tag{49}$$

$$\boldsymbol{R}_{\mathrm{O}\ell}^{\{d\}} = \begin{pmatrix}\boldsymbol{W}_{\mathrm{O}\ell}^{\{d\}} & \boldsymbol{o} & \boldsymbol{O}\\ \boldsymbol{o}^T & 1 & \boldsymbol{o}^T\\ \boldsymbol{O} & \boldsymbol{o} & \boldsymbol{U}_{\mathrm{O}\ell}^{\{d\}}\end{pmatrix} \tag{50}$$

$$\boldsymbol{Q}_{\mathrm{E}}^{\{d\}}(\boldsymbol{z}) = \boldsymbol{B}_M \boldsymbol{\Lambda}_{\mathrm{E}}^{\{d\}}(\boldsymbol{z}) \boldsymbol{B}_M \tag{51}$$

$$\boldsymbol{Q}_{\mathrm{O}}^{\{d\}}(\boldsymbol{z}) = \boldsymbol{B}_M \boldsymbol{\Lambda}_{\mathrm{O}}^{\{d\}}(\boldsymbol{z}) \boldsymbol{B}_M \tag{52}$$

where $\boldsymbol{W}_{\mathrm{E}\ell}^{\{d\}}$ is an $\lceil M/2 \rceil \times \lceil M/2 \rceil$ orthonormal matrix, and $\boldsymbol{W}_{\mathrm{O}\ell}^{\{d\}}, \boldsymbol{U}_{\mathrm{E}\ell}^{\{d\}}$ and $\boldsymbol{U}_{\mathrm{O}\ell}^{\{d\}}$ are $\lfloor M/2 \rfloor \times \lfloor M/2 \rfloor$ orthonormal matrices. In addition

$$\boldsymbol{\Lambda}_{\mathrm{E}}^{\{d\}}(\boldsymbol{z}) = \begin{pmatrix} \boldsymbol{I}_{\lceil M/2 \rceil} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{z}^{-\mathbf{1}^{\{d\}}} \boldsymbol{I}_{\lfloor M/2 \rfloor} \end{pmatrix} \tag{53}$$

$$\boldsymbol{\Lambda}_{\mathrm{O}}^{\{d\}}(\boldsymbol{z}) = \begin{pmatrix} \boldsymbol{I}_{\lfloor M/2 \rfloor} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{z}^{-\mathbf{1}^{\{d\}}} \boldsymbol{I}_{\lceil M/2 \rceil} \end{pmatrix}. \tag{54}$$

By controlling the matrices $\boldsymbol{W}_{\mathrm{E}\ell}^{\{d\}}$, $\boldsymbol{W}_{\mathrm{O}\ell}^{\{d\}}$, $\boldsymbol{U}_{\mathrm{E}\ell}^{\{d\}}$, and $\boldsymbol{U}_{\mathrm{O}\ell}^{\{d\}}$, we can design MD-LPPUFB's with the guarantee of the PU and LP properties. Since these matrices are orthonormal matrices and can be characterized by a combination of Givens rotations [1]–[3], using an unconstrained nonlinear optimization process for designing such systems is possible. In the case of $D = 1$, (48) is reduced to the factorization of an odd number of channel 1-D LPPUFB's [10], [11].

Equation (48) is obtained in a similar way to the approach that was done for even M since the following order-increasing process holds both of the PU and LP properties.

$$\boldsymbol{E}_{2(\ell+1)}(\boldsymbol{z}) = \boldsymbol{R}_{\mathrm{E},\ell+1}^{\{d\}} \boldsymbol{Q}_{\mathrm{E}}^{\{d\}}(\boldsymbol{z}) \boldsymbol{R}_{\mathrm{O},\ell+1}^{\{d\}} \boldsymbol{Q}_{\mathrm{O}}^{\{d\}}(\boldsymbol{z}) \boldsymbol{E}_{2\ell}(\boldsymbol{z}) \tag{55}$$

where $\boldsymbol{E}_m(\boldsymbol{z})$ denotes a polyphase matrix of LPPUFB's, whose dth-dimension order is m. When $\boldsymbol{E}_{2\ell}(\boldsymbol{z})$ is PU, the PU property of $\boldsymbol{E}_{2(\ell+1)}(\boldsymbol{z})$ is obvious. Thus, let us verify only the LP condition.

Equation (55) gives

$$\boldsymbol{E}_{2\ell}(\boldsymbol{z}) = \boldsymbol{Q}_{\mathrm{O}}^{\{d\}}(\boldsymbol{z}^{-\boldsymbol{I}}) \boldsymbol{R}_{\mathrm{O},\ell+1}^{\{d\}T} \boldsymbol{Q}_{\mathrm{E}}^{\{d\}}(\boldsymbol{z}^{-\boldsymbol{I}}) \boldsymbol{R}_{\mathrm{E},\ell+1}^{\{d\}T} \boldsymbol{E}_{2(\ell+1)}(\boldsymbol{z}). \tag{56}$$

Substituting this into the LP condition of $\boldsymbol{E}_{2\ell}(\boldsymbol{z})$, that is, $\boldsymbol{E}_{2\ell}(\boldsymbol{z}) = \boldsymbol{z}^{-2\boldsymbol{c}_{\Xi_{2\ell}}} \boldsymbol{D}_M \boldsymbol{E}_{2\ell}(\boldsymbol{z}^{-\boldsymbol{I}}) \boldsymbol{J}_M$, we have

$$\begin{aligned} &\boldsymbol{Q}_{\mathrm{O}}^{\{d\}}(\boldsymbol{z}^{-\boldsymbol{I}}) \boldsymbol{R}_{\mathrm{O},\ell+1}^{\{d\}T} \boldsymbol{Q}_{\mathrm{E}}^{\{d\}}(\boldsymbol{z}^{-\boldsymbol{I}}) \boldsymbol{R}_{\mathrm{E},\ell+1}^{\{d\}T} \boldsymbol{E}_{2(\ell+1)}(\boldsymbol{z}) \\ &\quad = \boldsymbol{z}^{-2\boldsymbol{c}_{\Xi_{2\ell}}} \boldsymbol{D}_M \boldsymbol{Q}_{\mathrm{O}}^{\{d\}}(\boldsymbol{z}) \boldsymbol{R}_{\mathrm{O},\ell+1}^{\{d\}T} \boldsymbol{Q}_{\mathrm{E}}^{\{d\}}(\boldsymbol{z}) \\ &\qquad \cdot \boldsymbol{R}_{\mathrm{E},\ell+1}^{\{d\}T} \boldsymbol{E}_{2(\ell+1)}(\boldsymbol{z}^{-\boldsymbol{I}}) \boldsymbol{J}_M \end{aligned} \tag{57}$$

where $\boldsymbol{c}_{\Xi_m}$ is the center of $\mathcal{N}(\boldsymbol{\Xi}_m)$, and $\boldsymbol{\Xi}_m$ is an extension matrix whose dth diagonal element is $m+1$.

From the facts that

$$\boldsymbol{R}_{\mathrm{O}m}^{\{d\}} \boldsymbol{Q}_{\mathrm{O}}^{\{d\}}(\boldsymbol{z}) \boldsymbol{D}_M \boldsymbol{Q}_{\mathrm{O}}^{\{d\}}(\boldsymbol{z}) \boldsymbol{R}_{\mathrm{O}m}^{\{d\}T} = \boldsymbol{z}^{-\mathbf{1}^{\{d\}}} \boldsymbol{\Delta}^{\{d\}}(\boldsymbol{z}) \boldsymbol{D}_M \tag{58}$$

$$\boldsymbol{R}_{\mathrm{E}m}^{\{d\}} \boldsymbol{Q}_{\mathrm{E}}^{\{d\}}(\boldsymbol{z}) \boldsymbol{\Delta}^{\{d\}}(\boldsymbol{z}) \boldsymbol{D}_M \boldsymbol{Q}_{\mathrm{E}}^{\{d\}}(\boldsymbol{z}) \boldsymbol{R}_{\mathrm{E}m}^{\{d\}T} = \boldsymbol{z}^{-\mathbf{1}^{\{d\}}} \boldsymbol{D}_M \tag{59}$$

where

$$\boldsymbol{\Delta}^{\{d\}}(\boldsymbol{z}) = \begin{pmatrix} \boldsymbol{I}_{\lfloor M/2 \rfloor} & \boldsymbol{o} & \boldsymbol{O} \\ \boldsymbol{o}^T & \boldsymbol{z}^{-\mathbf{1}^{\{d\}}} & \boldsymbol{o}^T \\ \boldsymbol{O} & \boldsymbol{o} & \boldsymbol{I}_{\lfloor M/2 \rfloor} \end{pmatrix} \tag{60}$$

(57) can be reduced to

$$\boldsymbol{E}_{2(\ell+1)}(\boldsymbol{z}) = \boldsymbol{z}^{-2\boldsymbol{c}_{\Xi_{2(\ell+1)}}} \boldsymbol{D}_M \boldsymbol{E}_{2(\ell+1)}(\boldsymbol{z}^{-\boldsymbol{I}}) \boldsymbol{J}_M \tag{61}$$

where $2\boldsymbol{c}_{\Xi_{2(\ell+1)}} = 2\boldsymbol{c}_{\Xi_{2\ell}} + 2\mathbf{1}^{\{d\}}$, namely, the dth-dimension order $[2\boldsymbol{c}_{\Xi_{2(\ell+1)}}]_d$ equals $2(\ell+1)$. As a result, the dth-dimension order is increased from 2ℓ to $2(\ell+1)$, holding the LP property.

TABLE I

CODING GAIN G_{SBC} OF MD-LPPUFB'S WITH RECTANGULAR DECIMATION FOR THE ISOTROPIC acf MODEL WITH $\rho = 0.95$. $\boldsymbol{M}$ AND $\overline{\boldsymbol{n}}$ DENOTE THE DECIMATION MATRIX AND THE ORDER OF POLYPHASE MATRIX, RESPECTIVELY. M DENOTES THE NUMBER OF CHANNELS

$\mathbf{M}$	M	$\bar{\mathbf{n}}^T$	G_{SBC} [dB] SEPARABLE	G_{SBC} [dB] PROPOSED
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	4	$(0,0)$	8.12	8.12
		$(1,1)$	8.12	8.16
		$(2,2)$	8.12	8.88
$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$	9	$(0,0)$	9.98	9.99
		$(1,1)$	-	-
		$(2,2)$	9.98	10.77
$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	16	$(0,0)$	10.75	10.78
		$(1,1)$	11.20	11.28
		$(2,2)$	11.42	11.55

C. Minimality

A structure is said to be *minimal* if it uses the minimum number of delay elements for its implementation [1]. For a 1-D causal PU system $\boldsymbol{E}(z)$, it is known that $\deg(\det(\boldsymbol{E}(z))) = \deg(\boldsymbol{E}(z))$, where $\deg(\boldsymbol{H}(z))$ denotes the degree of $\boldsymbol{H}(z)$, that is, the minimum number of delay elements required to implement $\boldsymbol{H}(z)$.

Now, let us investigate the degree of our proposed structure. Note that the degree in terms of the dth-dimension delay element z_d^{-1} cannot be increased by choosing any value of delay elements of the other dimensions. Therefore

$$\deg^{\{d\}}(\boldsymbol{E}(\boldsymbol{z})) \geq \deg^{\{d\}}(\boldsymbol{E}(\boldsymbol{z}^{\{d\}})) \qquad d \in \{0, 1, \cdots, D-1\} \tag{62}$$

holds, where $\deg^{\{d\}}(\boldsymbol{E}(\boldsymbol{z}))$ denotes the degree in terms of z_d^{-1}, and $\boldsymbol{z}^{\{d\}}$ is a $D \times 1$ vector whose elements are all '1' except for the dth-variable element z_d, that is, $\boldsymbol{z}^{\{d\}} = z_d \mathbf{1}^{\{d\}} + \mathbf{1}^{\{\overline{d}\}}$. For example, $\boldsymbol{z}^{\{1\}} = (1, z_1, 1)^T$ in three dimensions.

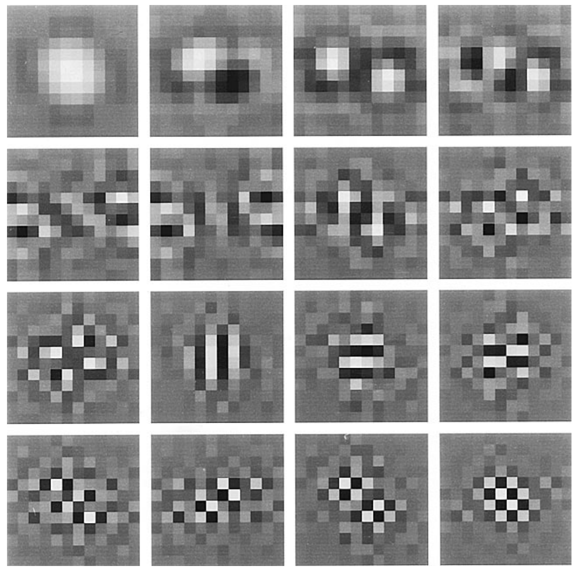
It can be verified that our proposed structure has

$$\deg^{\{d\}}(\det(\boldsymbol{E}(\boldsymbol{z}^{\{d\}}))) = \frac{N_d M}{2}, \qquad d \in \{0, 1, \cdots, D-1\}. \tag{63}$$

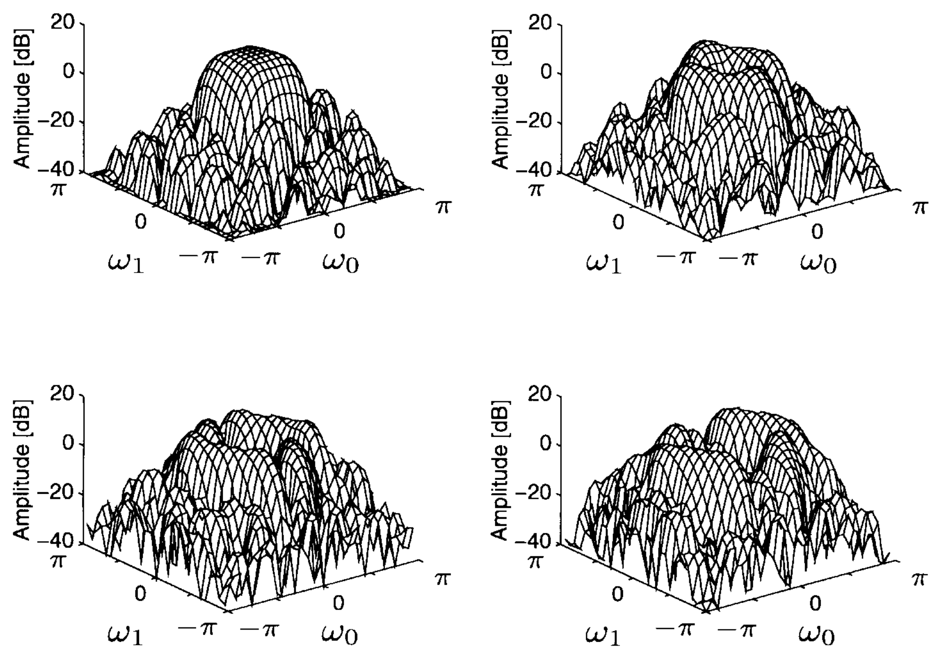
This implies that $\deg^{\{d\}}(\boldsymbol{E}(\boldsymbol{z}^{\{d\}})) = N_d M/2$ for $d \in \{0, 1, \cdots, D-1\}$ since $\boldsymbol{E}(\boldsymbol{z}^{\{d\}})$ can be regarded as a 1-D causal PU system $\boldsymbol{E}(z_d)$. Consequently, we have

$$\deg^{\{d\}}(\boldsymbol{E}(\boldsymbol{z})) \geq \frac{N_d M}{2}, \qquad d \in \{0, 1, \cdots, D-1\}. \tag{64}$$

The last inequality shows that our structure is minimal since it uses $N_d M/2$ delay elements for each dimension for its implementation. (Note that in the structure shown in Fig. 4, the downsamplers can be moved to the left side of the matrix $\boldsymbol{E}_0$ by using the noble identity [1], so that it is minimally implemented.)



(a)



(b)

Fig. 5. Design example of MD-LPPUFB's with rectangular decimation, where $\boldsymbol{M} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, $\varXi = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, and $\overline{\boldsymbol{n}} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Each filter has 12 × 12 taps. (a) Basis images of 16 analysis filters. (b) Amplitude responses of the four analysis filters whose subband signals have the four highest variances. $G_{\mathrm{SBC}} = 11.55$ dB for the isotropic acf model with $\rho = 0.95$. ω_d denotes the dth-dimension normalized angular frequency [rad].

D. No DC Leakage

When applied to subband image coding, filter banks should have bandpass and highpass filters that have *no DC leakage* [3]. This is because the DC leakage causes undesirable distortion in the reconstructed images when the subband signals are severely quantized.

The no-DC-leakage condition in the MD analysis bank is expressed as

$$\boldsymbol{h}(\boldsymbol{1}) = \boldsymbol{E}(\boldsymbol{1})\boldsymbol{d}_{\boldsymbol{M}}(\boldsymbol{1}) = (\sqrt{M} \quad 0 \quad 0 \quad \cdots \quad 0)^T \tag{65}$$

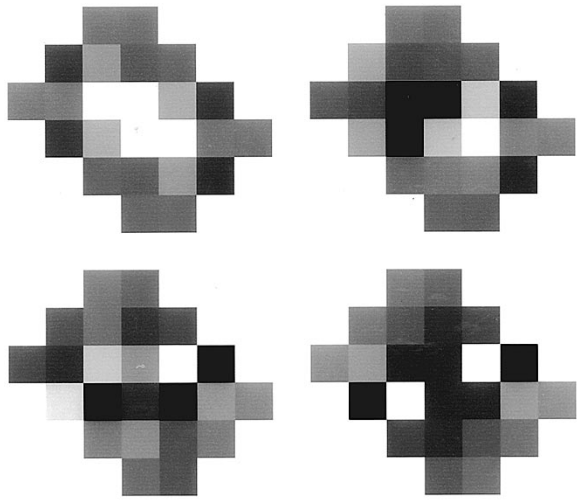
where $\boldsymbol{d}_{\boldsymbol{M}}(\boldsymbol{1})$ is the $M \times 1$ vector whose elements are all '1.'

Suppose that $\boldsymbol{E}_0\boldsymbol{d}_{\boldsymbol{M}}(\boldsymbol{1}) = (\sqrt{M}, 0, 0, \cdots, 0)^T$. In the proposed structure, the above condition can be reduced to

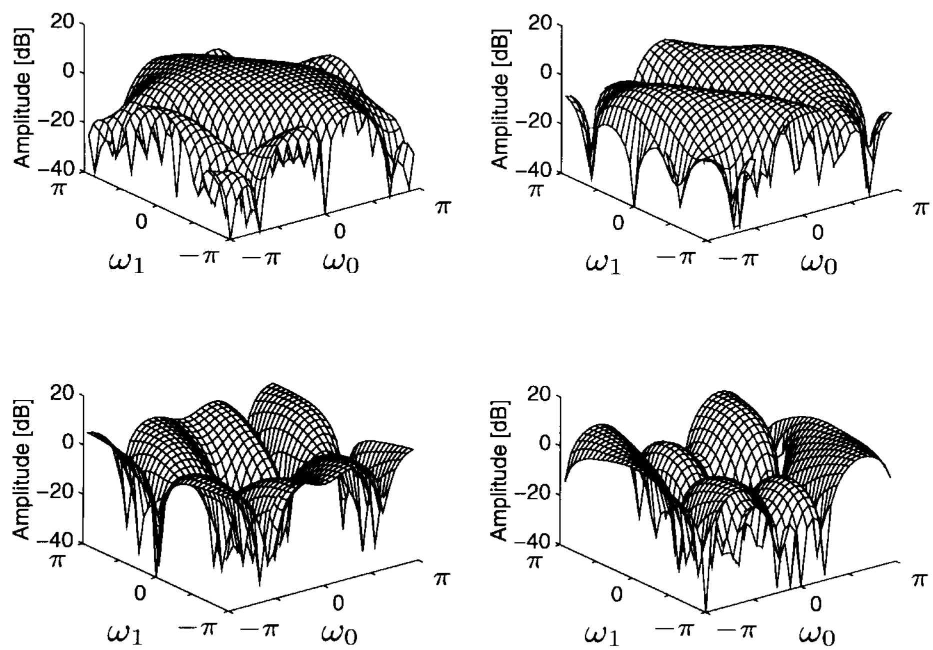
$$\left[\prod_{d=0}^{D-1} \prod_{\substack{n=1 \\ N_d \neq 0}}^{N_d} \boldsymbol{W}_n^{\{d\}}\right] \boldsymbol{W}_0^{\{\emptyset\}} = \begin{pmatrix} 1 & \boldsymbol{o}^T \\ \boldsymbol{o} & \boldsymbol{V} \end{pmatrix} \tag{66}$$

for even M or

$$\left[\prod_{d=0}^{D-1} \prod_{\substack{\ell=1 \\ L_d \neq 0}}^{L_d} \left\{ \boldsymbol{W}_{\mathrm{E}\ell}^{\{d\}} \begin{pmatrix} \boldsymbol{W}_{\mathrm{O}\ell}^{\{d\}} & \boldsymbol{o} \\ \boldsymbol{o}^T & 1 \end{pmatrix} \right\}\right] \boldsymbol{W}_{\mathrm{E}0}^{\{\emptyset\}} = \begin{pmatrix} 1 & \boldsymbol{o}^T \\ \boldsymbol{o} & \boldsymbol{V} \end{pmatrix} \tag{67}$$



(a)



(b)

Fig. 6. Design example of MD-LPPUFB's with nonrectangular decimation, which is designed under the no-DC-leakage condition, where $\boldsymbol{M} = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$, $\Xi = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, and $\bar{\boldsymbol{n}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Each filter has 24 taps. (a) Basis images of four analysis filters. (b) Amplitude responses of four analysis filters. $G_{\mathrm{SBC}} = 8.46$ dB for the isotropic acf model with $\rho = 0.95$. ω_d denotes the dth-dimension normalized angular frequency [rad].

for odd M, where $\boldsymbol{V}$ is a $(\lceil M/2 \rceil - 1) \times (\lceil M/2 \rceil - 1)$ orthonormal matrix. The above condition is easily derived from the facts that $\boldsymbol{Q}^{\{d\}}(\boldsymbol{1}) = \boldsymbol{I}, \boldsymbol{Q}_{\mathrm{E}}^{\{d\}}(\boldsymbol{1}) = \boldsymbol{I}$, and $\boldsymbol{Q}_{\mathrm{O}}^{\{d\}}(\boldsymbol{1}) = \boldsymbol{I}$.

For even M, a design made by controlling the matrices $\boldsymbol{R}_n^{\{d\}}$ in (36) subject to (66) leads to MD-LPPUFB's that have no DC leakage. The design can be achieved by restricting the matrix $\boldsymbol{W}_0^{\{\emptyset\}}$ to a matrix whose first column vector is the transposition of the first row vector of the product $[\prod_{d=0}^{D-1} \prod_{\substack{n=1 \\ N_d \neq 0}}^{N_d} \boldsymbol{W}_n^{\{d\}}]$. Note that the inverse of the product is a candidate of $\boldsymbol{W}_0^{\{\emptyset\}}$, yielding no DC leakage. For odd M, a design made by controlling the matrices $\boldsymbol{R}_{\mathrm{E}\ell}^{\{d\}}$ and $\boldsymbol{R}_{\mathrm{O}\ell}^{\{d\}}$ in (49) and (50) subject to (67) leads to MD-LPPUFB's without DC leakage. Similarly, this design can be achieved by properly choosing the matrix $\boldsymbol{W}_{\mathrm{E}0}^{\{\emptyset\}}$.

We will show a design example with no DC leakage in the next section.

V. Design Examples

In order to verify the significance of our proposed structure, let us show some design examples for both of the rectangular and nonrectangular decimation cases. These examples are designed by taking the Givens rotation angles, which appear in the factorization of the orthonormal matrices controlled during optimization, as the design parameters, and by using the routine 'fminu' provided by the MATLAB optimization toolbox [17]. The sign parameters in the factorization of orthonormal matrices are heuristically determined. Although the examples shown here are in 2-D, our proposed structure is applicable to any dimension.

A. Rectangular Decimation

For the design examples shown here, the object function of the optimization is chosen as the maximum coding gain G_{SBC} [1], [5] for the isotropic autocorrelation function (acf) model, which is a representative nonseparable one with the correlation coefficient $\rho = 0.95$ [18]. The first matrix $\boldsymbol{E}_0$ is chosen to be the type-II 2-D DCT for even M and the type-I 2-D DCT for odd M [19], which can be rewritten as the form in (45).

Table I shows the resulting G_{SBC}'s of the proposed lattice structure. Those of separable structures with 1-D LPPUFB's are also shown [8], [10]. As an example, basis images and amplitude responses of analysis filters $H_k(\boldsymbol{z})$ generated with our proposed structure are given in Fig. 5, where $\boldsymbol{M} = \left(\begin{smallmatrix}4 & 0\\ 0 & 4\end{smallmatrix}\right)$ and $\boldsymbol{\Xi} = \left(\begin{smallmatrix}3 & 0\\ 0 & 3\end{smallmatrix}\right)$. The number of channels M is 16, and the number of taps is 12 × 12. In addition, the order of $\boldsymbol{E}(\boldsymbol{z})$ is $\bar{\boldsymbol{n}} = (N_0, N_1)^T = (2, 2)^T$.

From Table I, we notice that the G_{SBC} of our proposed structure is higher than that of the separable structures. In particular, when $\boldsymbol{M} = \left(\begin{smallmatrix}2 & 0\\ 0 & 2\end{smallmatrix}\right)$, the G_{SBC} of our proposed structure becomes higher as the order increases, whereas that of the two-channel-based separable system does not. This is because the separable system cannot have any overlapping solution, whereas our proposed structure can.

B. Nonrectangular Decimation

Let us now show a design example of a nonrectangular decimation case, with which a separable system cannot be constructed. The object function here is also chosen as the coding gain G_{SBC} for the isotropic acf model with $\rho = 0.95$.

As an example, we choose the decimation matrix $\boldsymbol{M}$ and the extension matrix $\boldsymbol{\Xi}$ as $\boldsymbol{M} = \left(\begin{smallmatrix}2 & 1\\ 2 & -1\end{smallmatrix}\right)$ and $\boldsymbol{\Xi} = \left(\begin{smallmatrix}2 & 3\\ 0 & 3\end{smallmatrix}\right)$, where the number of channels M is 4 and the number of taps of each filter is 24. In addition, the order of $\boldsymbol{E}(\boldsymbol{z})$ is $\bar{\boldsymbol{n}} = (N_0, N_1)^T = (1, 2)^T$. The structure shown in Fig. 4 corresponds to this example.

The basis images and the amplitude responses of the resulting analysis filters $H_k(\boldsymbol{z})$ are shown in Fig. 6, where the matrices $\boldsymbol{\Phi}_{\mathrm{S}}$ and $\boldsymbol{\Phi}_{\mathrm{A}}$ in the first matrix $\boldsymbol{E}_0$ are fixed as

$$\boldsymbol{\Phi}_{\mathrm{S}} = \boldsymbol{\Phi}_{\mathrm{A}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \tag{68}$$

This choice guarantees that $\boldsymbol{E}_0 \boldsymbol{d}_{\boldsymbol{M}}(\mathbf{1}) = (\sqrt{M}\ 0\ 0 \cdots 0)^T$. In addition, the matrix $\boldsymbol{W}_0^{\{\emptyset\}}$ is chosen as the inverse of the product $\boldsymbol{W}_2^{\{1\}} \boldsymbol{W}_1^{\{1\}} \boldsymbol{W}_1^{\{0\}}$ so that (66) holds, that is, no DC leakage is caused. In Table II, we give the resulting optimal matrices $\boldsymbol{W}_n^{\{d\}}$ and $\boldsymbol{U}_n^{\{d\}}$.

In this example, the coding gain results in a $G_{\mathrm{SBC}} = 8.46$ dB, whereas $G_{\mathrm{SBC}} = 8.47$ dB when optimizing the full structure without considering the no-DC-leakage condition. These results are comparable.

TABLE II
OPTIMAL MATRICES DESIGNED FOR MAXIMIZING THE CODING GAIN OF THE STRUCTURE SHOWN IN FIG. 4 FOR THE ISOTROPIC acf MODEL WITH $\rho = 0.95$

d	n	$\mathbf{W}_n^{\{d\}}$	$\mathbf{U}_n^{\{d\}}$
$\emptyset$	0	$\begin{pmatrix}-0.4116 & -0.9114\\ 0.9114 & -0.4116\end{pmatrix}$	$\begin{pmatrix}0.9999 & 0.0101\\ 0.0101 & -0.9999\end{pmatrix}$
0	1	$\begin{pmatrix}-0.1771 & 0.9842\\ -0.9842 & -0.1771\end{pmatrix}$	$\begin{pmatrix}0.1869 & 0.9824\\ -0.9824 & 0.1869\end{pmatrix}$
1	1	$\begin{pmatrix}0.9990 & -0.0456\\ 0.0456 & 0.9990\end{pmatrix}$	$\begin{pmatrix}-0.9994 & -0.0355\\ -0.0355 & 0.9994\end{pmatrix}$
1	2	$\begin{pmatrix}0.9577 & 0.2877\\ -0.2877 & 0.9577\end{pmatrix}$	$\begin{pmatrix}0.9313 & -0.3642\\ -0.3642 & -0.9313\end{pmatrix}$

VI. CONCLUSIONS

We have proposed a lattice structure of MD-LPPUFB's. The lattice structure can produce MD-LPPUFB's whose filters all have the extended region of support $\mathcal{N}(\boldsymbol{M\Xi})$, where $\boldsymbol{M}$ is the decimation matrix, and $\boldsymbol{\Xi}$ is a positive integer diagonal matrix (or *extension matrix*) under the condition that $\mathcal{N}(\boldsymbol{M})$ is reflection invariant. Since the system structurally restricts both the PU and LP properties, an unconstrained optimization process can be used to design it. Our structure is developed for both an even and odd number of channels and includes the conventional 1-D system as a special case. It was also shown to be minimal, and the no-DC-leakage condition was presented. By showing some design examples, we verified the significance of the proposed structure for both the rectangular and nonrectangular decimation cases. For the rectangular decimation case, we showed that the structure achieves a higher coding gain for the isotropic acf model than that for the separable one. In particular, our proposed structure overcomes the problem of separable MD-LPPUFB's in that they cannot be constructed with any overlapping filters when they are based on two-channel 1-D systems. Furthermore, we demonstrated that the proposed lattice structure can generate a nonrectangular decimation LPPUFB with no DC leakage.

For future work, questions remain about the lattice structure. One question is whether the structure is complete for the class dealt with in this paper. The proof for the completeness of 1-D LPPUFB's shown in [6] and [11] might not be applicable to general MD-LPPUFB's since the proof requires us to solve an orthogonalization problem of multivariable polynominal matrices. Our conjecture is that our proposed structure is complete for degree-factorable LPPUFB's [14], [20], [21], although this has yet to be proven. Note that since degree-factorable filter banks are defined in two dimensions, the extension to multiple dimensions is relatively easy. Another question is how to avoid insignificant local minimal solutions in optimization processes. In addition, it is also of interest to investigate another class of MD-LPPUFB's provided by releasing the constraint on the region of support of filters.

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