Symmetric-extension Based Whitening for Phase Correlation

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Abstract—Symmetric-extension based whitening for phase correlation is proposed. The signs of DCT coefficients can be used for correlation calculation by this whitening. As a result, memory complexity and computational complexity are significantly saved. In addition, symmetric extension can attenuate the effect of discontinuity between endpoints of signals which is caused by discrete Fourier transform. Some experimental results show the effectiveness of symmetric-extension based whitening for phase correlation.

I. INTRODUCTION

Symmetric extension creates symmetry at the endpoints for smooth boundaries in finite-length signals [1]. It is particularly useful for image filtering and accordingly it has been mainly studied for convolution. Correlation, on the other hand, is widely used as a measure of similarity in many areas such as communications, pattern recognition, and cryptanalysis.

Phase correlation is a correlation method in which the spectral magnitude of signals is normalized [2][3]. This normalization effectively ‘whitens’ signals. The whitening provides effectiveness for alignment in addition to robustness against illumination variation, because an exact location can be estimated from the form of phase correlation function [4]. However, since the discrete Fourier transform (DFT) coefficients of signals are generally complex numbers, the cost of computational complexity and memory complexity is high. In addition, the discontinuity between endpoints affect phase correlation function strongly, which causes the failure of estimation. From the relationship between phase of DFT and signs of discrete cosine transform (DCT), DCT sign correlation is proposed by the authors [5][6], in which the effect of whitening is not considered.

In the present paper, we propose symmetric-extension based whitening for phase correlation. The effect for whitening is considered and the effect of extension can be reduced. In addition to the effect of whitening, proposed method is achieved with less computational complexity and memory complexity compared to phase correlation. Moreover, it can attenuate the effect of discontinuity caused by DFT.

II. PRELIMINARIES

In this section, general cross correlation, phase correlation, and DCT sign correlation are described. One dimensional expression is used for the sake of brevity. Z denotes the set of integer number.

A. General cross correlation

Let \( x(n), n \in \mathbb{Z} \), be \( N \)-point signal. The \( N \)-point DFT, \( X(k), k = 0, 1, \cdots, N-1 \) of \( x(n) \) is defined as

\[
X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}
\]

where \( W_N = \exp(-j2\pi/N) \) and \( j = \sqrt{-1} \).

General cross correlation function, \( r(n) \), of two \( N \)-point signals, \( x_1(n) \) and \( x_2(n) \) is defined as

\[
r(n) = \alpha \sum_{k=0}^{N-1} \frac{1}{W_1(k)}X_1^*(k)\frac{1}{W_2(k)}X_2(k)W_N^{-kn}
\]

where \( \alpha \) denotes the scale factor, \( W_1 \) and \( W_2 \) are the weights, and \( X_1^*(k) \) denotes the complex conjugate of \( X_1(k) \).

B. Phase correlation

\( X(k) \) is expressed in polar form in terms of its magnitude, \( |X(k)| \), and its phase factor, \( \phi_X(k) \), as

\[
X(k) = |X(k)|\phi_X(k).
\]

The phase correlation function, \( r_\phi(n) \), of two \( N \)-point signals, \( x_1(n) \) and \( x_2(n) \) is defined as

\[
r_\phi(n) = \frac{1}{N} \sum_{k=0}^{N-1} \phi_{x_1}^*(k)\phi_{x_2}(k)W_N^{-kn}
\]

where \( \text{If } |X(k)| = 0 \), then \( \phi_X(k) \) is set to 0 [2]. Therefore, the spectral magnitude is used as \( W_1(k) \) and \( W_2(k) \) in Eq. (2), which is referred to as ‘whitening’ in this paper.

C. DCT sign correlation

\( N \)-point DCT, \( X_C(k) \) for \( k = 0, 1, \cdots, N-1 \), of \( x(n) \) is defined as

\[
X_C(k) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} k_n x(n) \cos \left( \frac{\pi(n+1/2)k}{N} \right)
\]

where

\[
k_k = \begin{cases} 
1/\sqrt{2}, & k = 0 \\
1, & k \neq 0
\end{cases}
\]
DCT sign correlation, \( r_C(n) \), of two \( N \)-point signals, \( x_1(n) \) and \( x_2(n) \), is defined as

\[
r_C(n) = \frac{1}{N} \sum_{k=0}^{N-1} K_k \sigma_{x_1}(k) \sigma_{x_2}(k) \cos \left( \frac{\pi nk}{N} \right)
\]

(7)

where \( K_k \) is generally set to \( k^2 \), and \( \sigma_{x_1}(k) \) and \( \sigma_{x_2}(k) \) denote the sign of DCT coefficients \( X_{1C}(k) \) and \( X_{2C}(k) \), respectively [5].

**D. Correlation coefficient matrix**

Correlation coefficient, \( r_{XX}(l) \), of \( N \)-point signal \( x(n) \) is defined as

\[
r_{XX}(l) = \frac{\sum_{n=0}^{N-1} (x(n) - \bar{x})(x(n + l) - \bar{x})}{\sum_{n=0}^{N-1} (x(n) - \bar{x})^2}
\]

(8)

where \( l \) denotes a lag and \( \bar{x} = \sum_{n=0}^{N-1} x(n)/N \). Correlation between samples which are away from \( l \) on a signal is evaluated by correlation coefficient. It is well known that the correlation coefficients of natural images are approximated with \( \rho < 1 \) as

\[
r_{XX}(l) \simeq \rho^{|l|}.
\]

(9)

That is, the nearest samples are highly correlated.

The correlation coefficient matrix is defined as

\[
R_{XX} = RR^t
\]

(10)

where \( R^t \) denotes the transpose of \( R \) and

\[
R = [r_{XX}(0), r_{XX}(1), r_{XX}(2), \ldots, r_{XX}(M-1)]^t.
\]

(11)

The correlation coefficient matrix in AR(1) process is given as a Toeplitz matrix [7], i.e.,

\[
R_{XX} = RR^t = \begin{pmatrix}
1 & \rho & \rho^2 & \cdots & \rho^{M-1} \\
\rho & 1 & \rho & \cdots & \rho^{M-1} \\
\rho^2 & \rho & 1 & \cdots & \rho^{M-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho^{M-1} & \rho^{M-2} & \rho^{M-3} & \cdots & 1
\end{pmatrix}.
\]

(12)

**III. SYMMETRIC-EXTENSION BASED WHITENING**

Symmetric-extension based whitening for phase correlation is proposed and its effect of whitening is considered.

**A. Symmetrically extended signals and weights for whitening**

Symmetrically extended signal is defined as

\[
x_s(n) = \begin{cases} 
x(n), & n = 0, 1, \ldots, N - 1 \\
x(2N - n - 1), & n = N, N + 1, \ldots, 2N - 1
\end{cases}
\]

(13)

It is well known that the relationship between DFT coefficients, \( X_s(k) \) of \( x_s(n) \) and DCT coefficients \( X_C(k) \) of \( x(n) \), i.e., for \( k = 0, 1, \ldots, N - 1 \)

\[
X_C(k) = \sqrt{\frac{1}{2N}} k_k W_1(k) X_{1C}(k) W_2(k) X_{2C}(k)
\]

(14)

Therefore, we can handle the DCT coefficients of the original signal as the DFT coefficients of its symmetrically extended signal. That is, the length of a signal and its calculation for transform are not increased in order to obtain the DFT coefficients of the symmetrically extended signal.

We propose the use of the absolute value of DCT coefficients as a weight for whitening for symmetrically extended signals. In this condition, the signs of DCT coefficients can be used for correlation because the DFT coefficients of the symmetrically extended signals are calculated by DFT of the original signals, i.e.,

\[
r_s(n) = \frac{1}{N} \sum_{k=0}^{N-1} K_k W_1(k) X_{1C}(k) W_2(k) X_{2C}(k) \cos \left( \frac{\pi nk}{N} \right)
\]

(15)

where \( W_1(k) = |X_{1C}(k)| \) and \( W_2(k) = |X_{2C}(k)| \). That is, without increasing samples, we can obtain the effect of whitening with less computational complexity.
TABLE I
THE NUMBER OF OPERATIONS FOR CORRELATION BETWEEN SIGNALS OF LENGTH N

<table>
<thead>
<tr>
<th>method</th>
<th>multiplications</th>
<th>additions</th>
</tr>
</thead>
<tbody>
<tr>
<td>phase correlation</td>
<td>$3((N \log_2 N)/2 + N)$</td>
<td>$2((N \log_2 N) + 1)$</td>
</tr>
<tr>
<td>proposed</td>
<td>$3((N \log_2 N)/2 + N)$</td>
<td>$3((N \log_2 N)/2 + N)$</td>
</tr>
</tbody>
</table>

B. Effect of whitening

Figure 1 shows how much a sample is correlated to the one at $l$ samples away in the horizontal direction of Lena image. In the original signal, the nearer two samples are located, the more they are correlated to each other. Conversely, in the weighted signal, they are not correlated regardless of the location, which provides us the effect of whitening.

Figure 2 shows the correlation coefficient matrix, $R_{XX}$. In the original signal case (upper), $R_{XX}$ is approximated as the Toeplitz matrix with $\rho = 0.96$. Conversely, in symmetric-extension based whitening and whitening cases, $R_{XX}$ is approximated as identity matrix $I$. In symmetric-extension based whitening case (lower left), the mean and variance of the absolute error between $R_{XX}$ and $I$ are $0.0246$ and $6.44 \times 10^{-4}$, respectively. In whitening case (lower right), the mean and variance of the absolute error between them are $0.0263$ and $3.94 \times 10^{-4}$, respectively.

C. Computational complexity

The number of operations is summarized in Table I, which is calculated on the basis of the following.

Wang’s algorithm, which is a fast algorithm of DCT, achieves $N$-point DCT with

\[
\begin{align*}
\mu_N &= (N/2) \log_2 N + 1, \\
\alpha_N &= (3N/2) \log_2 N - N + 1
\end{align*}
\]

where $\mu_N$ and $\alpha_N$ denote the number of multiplications and additions for real numbers [8]. $N$-point FFT is achieved by

\[
\begin{align*}
M_N &= (N/2) \log_2 N, \\
A_N &= N \log_2 N
\end{align*}
\]

where $M_N$ and $A_N$ denote the number of multiplications and additions for complex numbers, respectively [9]. The multiplication for complex numbers corresponds to three times as many as that for real numbers [10]. The number of multiplication for phase correlation and proposed method is shown in Fig. 3 in which the signal length is 1 to 256.

IV. SIMULATIONS

A. Shift estimation for one dimensional expression

We performed shift estimation to show the effect of whitening for two cases, ‘not whitening’ and ‘whitening’, in the original signal and symmetrically extended signal. One line of Lena image was used for simulation. Figure 4 shows the result. In the case of nonprocessed signal (upper), the form of the correlation function in the original signal is different from that in symmetrically extended signal. Conversely, in the case of whitening (lower), the form of the correlation function in the original signal is similar to that in symmetrically extended signals.

B. Shift estimation for two dimensional expression

We also performed two dimensional version. Figure 8 shows correlation surfaces in three cases: the first one is that two input signals are used without weighting for whitening, the second one is that only one signal is weighted for whitening, the third one is that both signals are weighted for whitening. We can see that the peak form is sharpen by whitening.

C. Effect for smoothness

We performed shift estimation between two signals by phase correlation in the following four cases to demonstrate the effectiveness against discontinuity for the proposed method. The first case is that the signals were used without preprocessing, the second one is that the signals were used after windowing by Hann window, the third one is that the signals were used after decay extension [11], and the fourth one is that the signals were extended symmetrically.

Figure 6 shows the two signals, $x(n)$ of length 64 and $h(n)$ of length 32 of one line of Lena image, for correlation, in
which \( h(n) = x(n + n_0) \) and \( n_0 = 32 \). Figure 7 shows the results. In the fourth case, the shift was correctly estimated as shown in Fig. 7(d). In the other three cases, the shift was not correctly estimated. We also performed two-dimensional version. Figure 8 shows the phase correlation surfaces between two images, one is of size \( 64 \times 64 \), and the other is of size \( 32 \times 32 \) in Lena image. The shift was set to 8 in both the horizontal and vertical directions. In the original images (left), the estimation failed, while in the proposed method, the shift could be correctly estimated (right).

V. CONCLUSIONS

We have proposed symmetric-extension based whitening for phase correlation and its effect for whitening has been considered. The DFT coefficients of symmetric extension are obtained by DCT without increasing samples. The signs are used for correlation calculation by using the absolute value of DCT coefficients as a weight for whitening, which contributes to the computational complexity and memory complexity. In addition, symmetric extension attenuates the effect of discontinuity. Some experimental results have been demonstrated to show the appropriateness and effectiveness of symmetric-extension based whitening.

REFERENCES